CRASH COURSE FOR NET JRF/GATE

(MATHEMATICS)

Limited No. of Registration

"HURRY UP"

NAME..

SUBJECT.......Complex Analysis (Note)........

"Jai Maa Vaishno Devi"

I dedicate this note book to my Respected Teacher Mr. Rajendra Dubey (Director of DIPS Academy) who initiated how to learn Complex Analysis

Last but not the least, thanks to almighty God for giving us the talent to write and share the fruits of this wonderful gift with all friends.

"हमारे बाद महफिल में अफसाने बयाँ होंगे,
बहारें हमको ढूँढेंगी जाने हम कहाँ होंगे।"

(By. S.K. Rathore)

Fellowship Awarded by (CSIR JRF)

"Nothing is tough in this universe
As though comes easy when you do tough."

S.K. RATHORE

Syllabus :-

Unit-1 :- Functions of complex variables, Limit, continuity, differentiability, Singularities, Classification of singularities, Construction of analytic functions

Unit-2 :- Complex Integration

Arc, Paths, Jordan curve etc.
Definition of complex integral by Riemann, Standard estimation of complex integrals. Various complex integral inequalities and formulae, Cauchy theorem, Cauchy Gaursat, Morera, Liouville (V. Imp).

Unit-3 :- Meromorphic function, Argument Principal, Rouche's theorem, Hurwitz theorem, Counting of zeros & poles of meromorphic functions, Residue theorem.

:- Calculus of Residues.

Unit-4 :- Conformal mapping.

Unit-5 :- Power Series.

UNIT-I

S.K. RATHORE

Complex Number :- A complex number is defined as the ordered pair (x, y) of real numbers, $z = (x, y)$ satisfying the following rules for addition and multiplication.

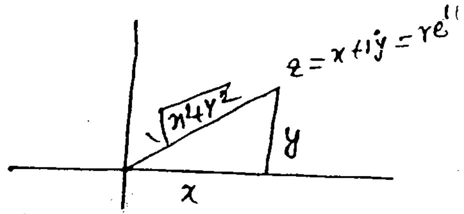
$$z_1 + z_2 = (x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$$

$$z_1 \cdot z_2 = (x_1, y_1) \cdot (x_2, y_2) = (x_1 x_2 - y_1 y_2, x_1 y_2 + x_2 y_1)$$

Every complex number can be written as

$$z = x + iy \quad \text{where } x, y \in \mathbb{R}$$

$$\text{or, } z = re^{i\theta}$$



Here $r = \sqrt{x^2+y^2}$

$\theta = \tan^{-1} \left| \frac{y}{x} \right|$

Every complex number can be represented geometrically as a point in xy-plane. We call this plane as complex plane or argand plane or argand diagram.

$|z|$ = Distance of z from origin '0' and we call it modulus of z.

$\arg z$ = The angle with the line joining the point with origin makes with +ve direction of x-axis.

The specific value of $\arg z$ satisfying $-\pi < \arg z \le \pi$ or $-\pi \le \arg z < \pi$ is called the principal value of $\arg z$ and we denote it by $\operatorname{Arg} z$.

Remarks :- There is no

→ meaning of graph in complex analysis.

→ Conjugate of complex numbers are always L.I.

→ [redacted]

V.imp. → Two complex numbers are equal iff their Principal Arg and modulus are same.

$$z = x+iy = \sqrt{x^2+y^2}\left(\frac{x}{\sqrt{x^2+y^2}} + i\frac{y}{\sqrt{x^2+y^2}}\right)$$

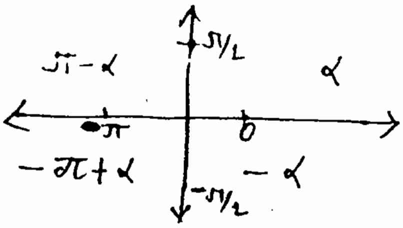
$$= re^{i\theta}$$

Here $r = \sqrt{x^2+y^2}$

$\theta = \tan^{-1}\left|\frac{y}{x}\right|$

If $-\pi < \theta \le \pi$ or $-\pi \le \theta < \pi$ then $\tan^{-1}\left|\frac{y}{x}\right|$ is called Principal argument of z.

$$\boxed{\arg z = \tan^{-1}\left(\frac{y}{x}\right) + 2n\pi}$$



$$\operatorname{Arg} z = \begin{cases} \alpha & a>0,\ b>0 \quad \text{1st quadrant} \\ \pi-\alpha & a<0,\ b>0 \quad \text{IInd quadrant} \\ -\pi+\alpha & a<0,\ b<0 \quad \text{IIIrd quadrant} \\ -\alpha & a>0,\ b<0 \quad \text{IVth quadrant} \\ 0 & a>0,\ b=0 \\ \pi & a<0,\ b=0 \\ \pi/2 & a=0,\ b>0 \\ -\pi/2 & a=0,\ b<0 \end{cases}$$

→ Let $z = x+iy$; $x>0,\ y>0$

$\alpha = \tan^{-1}\left|\frac{y}{x}\right|$; $0 \le \alpha < \pi/2$

e.g.:- $z = 1+\sqrt{3}i$

$$\alpha = \tan^{-1}\left|\frac{\sqrt{3}}{1}\right| = \tan^{-1}\sqrt{3} = \tan^{-1}(\tan \pi/3)$$

$$= \pi/3.$$

→ If $x<0,\ y>0$

e.g.: $z = -1+\sqrt{3}i$

$$\alpha = \tan^{-1}\left|\frac{\sqrt{3}}{-1}\right| = \tan^{-1}\left[\tan\left(\pi - \frac{\pi}{3}\right)\right] = \tan^{-1}\tan\left(\frac{2\pi}{3}\right)$$

$$= \frac{2\pi}{3}$$

Note: At origin $(0,0)$ [Here $x=0, y=0$] Arg z is not defined.

Change of Argument:

$$z = x+iy$$
$$|z| = \sqrt{x^2+y^2}$$
$$\arg z = \tan^{-1}\left(\frac{y}{x}\right) + 2n\pi$$

→ When a variable point z completes a circuit to occupy the same position & now if it is called z' then

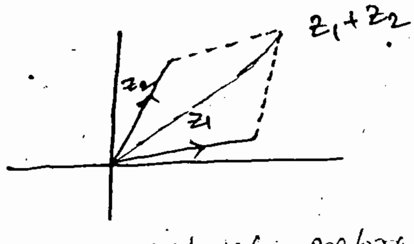
(i) $\arg z = \arg z'$
if the circuit does not encloses origin.

(ii) $\arg z + 2\pi = \arg z'$
if a circle encloses origin and we have direction is anticlockwise or counterclockwise (+ve).

→ $z_1 = (x_1, y_1) = x_1 + iy_1$; $z_2 = (x_2, y_2) = x_2 + iy_2$

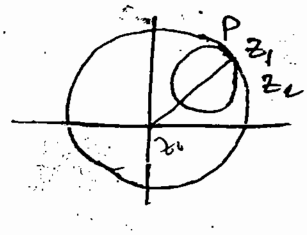
$|z_1 - z_2| =$ distance between the positions z_1 & z_2

$$= \sqrt{(x_1-x_2)^2 + (y_1-y_2)^2}$$



e.g.: $|z-a| + |z-b| = c$, which represents an ellipse.

Positional equality:- Let z be a variable point and initially at P in the complex plane and named z_1. When z moves around any point z_0 after one complete revolution it again arrives at P



and now let us call it z_2.

NOW. z_1 and z_2 are at the same position i.e. they have the same co-ordinate in the cartesian plane. Hence z_1 and z_2 are called positionally equal complex numbers.

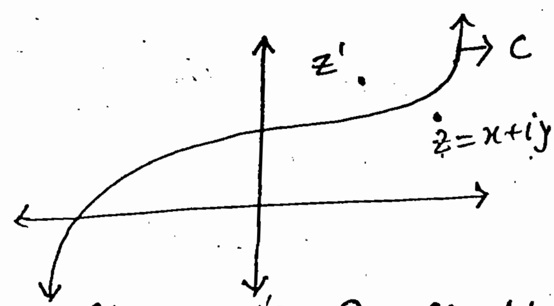
Note :- (i) While moving around z_0 the variable point z encloses origin then $\arg z_2$ is $\alpha + 2\pi$, where α is argument of z. After the nth revolution

$$\arg z_2 = \arg z + 2n\pi.$$

Hence though the numbers are positionally equal but they are considered different complex numbers.

(ii) When origin is not enclosed in the revolution then $\arg z_1 = \arg z_2$ after any number of complete revolution. Hence in this case complex numbers are equal.

Symmetry or Conjugacy :- Inverse Point



Let C be a curve in a complex plane & z be any point then the image of z taking curve as mirror is called symmetry or conjugate point of z w.r.t. the curve. Both z and z' are called conjugate of each other.

Self Conjugate :- A point is said to be self conjugate w.r.t. C iff it lies on the curve

i.e. $z = \bar{z}$.

$\rightarrow$ $\left.\begin{array}{l} z = x+iy \\ \bar{z} = x-iy \end{array}\right\}$ Conjugate w.r.t. x-axis.

Formula:- Let C be given by

$$C : |z - z_0| = r$$

and $a \in \mathbb{C}$. then the inverse of a w.r.t. C $= \boxed{z_0 + \dfrac{r^2}{\bar{a} - \bar{z}_0}}$

e.g.: (i) Inverse of $a = 2$ is ? when $C : |z| = 1$.

Solution:- Here $z_0 = 0$, $r = 1$, $a = 2$.

Inverse of 2 is $0 + \dfrac{1}{2-0} = \dfrac{1}{2}$ Ans

e.g. (ii) Find the inverse of $a = 1+i$ when $C : |z| = 1$.

Solution:- Here $z_0 = 0$, $r = 1$.

$$\text{Inverse of } (1+i) = 0 + \frac{1}{\overline{(1+i)} - 0}$$

$$= \frac{1}{1-i} \times \frac{1+i}{1+i}$$

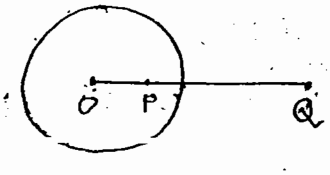
$$= \frac{1+i}{1+1} = \frac{1}{2}(1+i) \text{ Ans}$$

e.g. (iii) $\dfrac{|z-p|}{|z-q|} = k \neq 1$; p & q are inverse points.

This is a equation of circle.

Note:- Let P and Q are two points in complex plane and C be a circle with centre at O (not origin) then P and Q are said to be inverse point of each other if O, P, Q are collinear & $OP \cdot OQ = r^2$

The point lies on circumference of circle is self inverse.



Functions of complex variables :-

Let $f: \mathbb{C} \to \mathbb{C}$

$w = f(z) = u + iv$

where $z = x + iy$

$u = u(x,y)$

$v = v(x,y)$

i.e. A map $f: D \to \mathbb{C}$ where $D \subseteq \mathbb{C}$

s.t. $\boxed{w = f(z) = u(x,y) + iv(x,y)}$

Where f assigns every element of D to some element of $\mathbb{C}$, [redacted] be unique, is called the function of complex variables.

e.g. :- (i) $f(z) = z^2 = (x+iy)^2$

$$= x^2 - y^2 + 2xyi$$

$\Rightarrow \quad u = x^2 - y^2 \; ; \quad v = 2xy$

(ii) $f(z) = \log z$

$f(z) = \log z = \log r + i(\theta + 2k\pi)$

$$= \frac{1}{2}\log(x^2+y^2) + i\left[\tan^{-1}\left(\frac{y}{x}\right) + 2k\pi\right]$$

$\Rightarrow \quad u = \frac{1}{2}\log(x^2+y^2) \; ; \quad v = \tan^{-1}\left(\frac{y}{x}\right) + 2k\pi$

Type of complex valued function :- If $f: \mathbb{C} \to \mathbb{C}$ is a complex valued function

we define $w = f(z)$, where $z = x + iy$; $x,y \in \mathbb{R}$

i.e. z is trace of (x,y)

Similarly, we define $\bar{z} = x - iy$ and is trace of $(x,-y)$

Now, $w = f(z) = f(x+iy)$

$w = u(x,y) + iv(x,y)$

where u, v are real valued function of x,y.

i.e. $\boxed{w = u + iv}$

→ z and $\bar{z}$ are independent.

$\boxed{\text{Warning}}$:- $w = f(z)$ has no graphical representation
i.e. limit, continuity, differentiability etc. can not be verified with the help of graphical approach.
i.e. all the graphical analysis which were available for $y = f(x)$ are no longer in use for $w = f(z)$.

$\boxed{\text{e.g}}$:- $w = f(z) = \dfrac{1}{z} = \dfrac{\bar{z}}{|z|^2} = \dfrac{x}{x^2+y^2} - i\dfrac{y}{x^2+y^2}$

Here $u = \dfrac{x}{x^2+y^2}$, $v = \dfrac{-y}{x^2+y^2}$

$\Rightarrow u^2+v^2 = \dfrac{1}{x^2+y^2} \quad \Rightarrow x^2+y^2 = \dfrac{1}{u^2+v^2}$

Then $x = \dfrac{u}{u^2+v^2}$, $y = -\dfrac{v}{u^2+v^2}$

→ Given now,

$\boxed{a(x^2+y^2) + bx + cy + d = 0}$ —— (I)

then $a\left(\dfrac{1}{u^2+v^2}\right) + \dfrac{bu}{u^2+v^2} + \dfrac{cv}{u^2+v^2} + d = 0$

or, $\boxed{d(u^2+v^2) + bu - cv + a = 0}$ —— (II)

- If $a = 0$ then equ^n (I) is line equ^n and equ^n (ii) is circle passing through origin.
- If $d = 0$ then equ^n (I) is circle passing through origin but equ^n (ii) is line equ^n

$\boxed{\text{e.g}}$:- $y = 2x+1$ or $2x - y + 1 = 0$ is a line

$a=0,\ b=2,\ c=-1,\ d=1$

$(u^2+v^2) + 2u + v = 0 \Rightarrow \boxed{u^2+v^2+2u+v=0}$ ~~convert into~~ circle.

Note:- When a argument is taken from a curve in z-plane then the function $f(z)$ transform this curve to another plane which is plotted in some other plane w-plane whose real axis is u and imaginary axis is v.

Problem:- Show that $\dfrac{|z-p|}{|z-q|} = k$ $(k>0)$ is equation of circle or line where p and q are conjugate points w.r.t. the given circle or line.

Solution:- $|z-p| = k|z-q|$

If $k=1$ then it represents a straight line w.r.t. which p and q are symmetrical points.

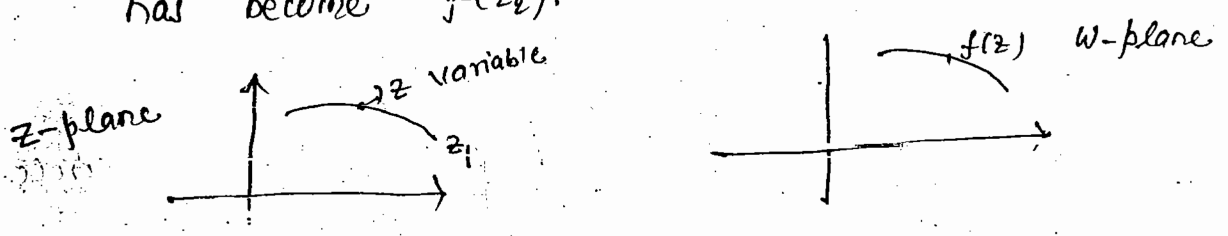
Otherwise if $k \neq 1$ then it represents a family of circles for every member of which p, q are inverse points.

Type of functions:-

One-one or Univalent:- If $f(z_1) = f(z_2) \Rightarrow z_1 = z_2$ then f is univalent otherwise <u>many valent</u>.

Single Valued function:- If $z_1 = z_2 \Rightarrow f(z_1) = f(z_2)$ then f is a single valued function.

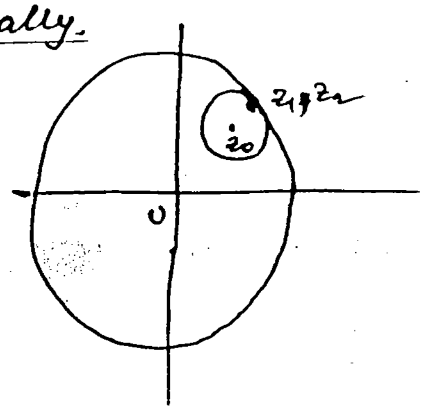
Here the equality between z_1 and z_2 and $f(z_1)$ & $f(z_2)$ is positional equality i.e. whenever z_1 occupies the original position after completing a circuit to become z_2, $f(z_1)$ should occupy the original position even if it has become $f(z_2)$.



Multivalued function:- $f: \mathbb{C} \to \mathbb{C}$

The function $w = f(z)$ is said to be multivalued if $\exists$ a point z_0 s.t. whenever a variable point z becomes z_2 from z_1 after one complete revolution around z_0, $f(z)$ becomes $f(z_2)$ from $f(z_1)$ but does not occupy the position of $f(z_1)$. Such point z_0 is called branch point.

i.e. $z_1 = z_2$ but $f(z_1) \neq f(z_2)$ positionally.



e.g.: (i) $f(z) = \sqrt{z} = z^{1/2} = \sqrt{r}\, e^{i\theta/2}$

$$z = re^{i\theta}$$

$$f(z_1) = \sqrt{r}\, e^{i\theta/2}$$

$$f(z_2) = \sqrt{r}\sqrt{e^{i(\theta+2\pi)}}$$

$$= \sqrt{r}\, e^{i\times\frac{1}{2}(\theta+2\pi)}$$

$$= \sqrt{r}\,.\, e^{\frac{i\theta}{2}}\,.\, e^{i\pi}$$

$$= -\sqrt{r}\, e^{i\theta/2} \qquad [\because e^{i\pi} = -1]$$

Hence origin is B.P. and function is multivalued.

(ii) $f(z) = \sqrt{z-a}$

If z moves around 'a' then w moves around origin.

$$z - a = re^{i\theta}$$

$\Rightarrow z = a$ is B.P.

$$z_1 = a + re^{i\theta}$$

$$z_2 = a + re^{i(\theta+2\pi)}$$

$$f(z_1) \neq f(z_2).$$

Branch cut :- Let $z=a$ is a branch point for $f(z)$. Then any line through 'a' is said to be branch cut if for any revolution around 'a', we do not agree to cross this line.

Note :- $f(z)$ may occupy the same position even if z moves around branch point.

e.g.: $f(z) = \sqrt{z}$

Then branch cuts are along $y=mx$ (since origin is the B.P)

→ Branch cuts are infinite for same B.P.

Some class of Multivalued functions :-

Functions	Possible Branch points
i). $\sqrt{z}$ or $z^{1/n}$, $n \in \mathbb{N}-\{1\}$ →	0
ii). $\log z$ →	0
iii). $\sin^{-1} z$ →	± 1
iv). $\cos^{-1} z$ →	± 1
v). $\tan^{-1} z$ →	$\pm i$
vi). $\sqrt{f(z)}$ or $[f(z)]^{1/n}$ →	roots of $f(z)=0$
vii). $\log f(z)$ →	roots of $f(z)=0$
viii). $\sin^{-1} f(z)$ →	roots of $f(z)\pm 1=0$
ix). $\cos^{-1} f(z)$, inverse function →	roots of $f(z)\pm 1=0$
x). $\sinh^{-1} f(z)$, inverse hyperbolic →	roots of $f(z)\pm i=0$
xi). $\tan^{-1} f(z)$ →	roots of $f(z)\pm i=0$

Note:- Method for finding Branch point :-

(i) $\sin^{-1}z$

$$\frac{d}{dz}\sin^{-1}z = \frac{1}{\sqrt{1-z^2}}$$

$$\sqrt{1-z^2} = 0$$

$$\Rightarrow 1-z^2 = 0$$

$$\Rightarrow z^2 = 1$$

$$\Rightarrow z = \pm 1$$

Hence 1 & -1 are branch points.

(ii) $\cos^{-1}z$

$$\frac{d}{dz}\cos^{-1}z = \frac{-1}{\sqrt{1-z^2}}$$

$$\sqrt{1-z^2} = 0$$

$$\Rightarrow 1-z^2=0$$

$$\Rightarrow z^2 = 1$$

$$\Rightarrow z = \pm 1$$

Hence 1 & -1 are branch points.

(iii) $\tan^{-1}z$

$$\frac{d}{dz}\tan^{-1}z = \frac{1}{1+z^2}$$

$$1+z^2=0 \Rightarrow z = \pm i$$

Hence i & $-i$ are branch points.

Note: This method is not correct because differentiation of multivalued function is not possible.

We can apply limit, continuity, differentiability only on single valued functions.

Kasana

Note: B.P. always lies on the real-axis.

Problem :- Find the branch point of $f(z) = \sqrt{\frac{z}{z-1}}$

Solution :- $f(z) = \sqrt{\frac{z}{z-1}}$

$$f(z_1) = \sqrt{\frac{re^{i\alpha}}{re^{i\alpha}-1}}$$

$$f(z_2) = \sqrt{\frac{re^{i(\alpha+2\pi)}}{re^{i(\alpha+2\pi)}-1}}$$

$$f(z_1) = -\sqrt{\frac{re^{i\alpha}}{re^{i\alpha}-1}}$$

$\Rightarrow$ 0 is the B.P. of $f(z)$

Also, let $z-1 = re^{i\alpha}$

$$\Rightarrow z = 1 + re^{i\alpha}$$

$$f(z_1) = \frac{\sqrt{1+re^{i\alpha}}}{\sqrt{re^{i\alpha}}}$$

$$f(z_2) = \frac{\sqrt{1+re^{i(\alpha+2\pi)}}}{\sqrt{re^{i(\alpha+2\pi)}}}$$

$$= \frac{\sqrt{1+re^{i\alpha}}}{-\sqrt{re^{i\alpha}}}$$

$$f(z_1) \neq f(z_2)$$

$\Rightarrow$ 1 is the B.P. of $f(z)$

Hence 0 & 1 are the branch pts. of $f(z) = \sqrt{\frac{z}{z-1}}$

$\rightarrow$ $f(z) = \log z$

$f(z_1) = \log(re^{i\alpha}) = \log r + i\alpha$

$f(z_2) = \log(re^{i(\alpha+2\pi)})$

$= \log r + i(\alpha+2\pi)$

$f(z_1) \neq f(z_2)$

$\Rightarrow$ $f(z) = \log z$ is multivalued function or infinite valued function.

$\boxed{z=0}$ is the B.P.

→ $f(z) = \sqrt{z}$ is two valued function because after two revolutions it occupies the same position.

→ $f(z) = z^{1/n}$ is n-valued function because after n revolutions it occupies the same position.

Problem:- Find the number of branch points of $f(z) = \tan^{-1}(2z + z^2)$

Solution:- Since the branch points of $\tan^{-1} f(z)$ is given by

$$f(z) \pm i = 0$$

$$\Rightarrow z^2 + 2z \pm i = 0$$

$$\Rightarrow z^2 + 2z + i = 0 \quad \text{or} \quad z^2 + 2z - i = 0$$

$z^2+2z+i=0$ ↓ It gives two B. Pts.

$z^2+2z-i=0$ ↓ It gives two B. Pts.

Hence $f(z) = \tan^{-1}(2z + z^2)$ has 4 branch points.

Note:- (i) $f(z) = a^z$
$f(z) = e^{z \log a}$ is a multivalued function if $a \in \mathbb{C} - \mathbb{R}^+$.

e.g.:- $f(z) = (-2)^z$
$= (-1)^z 2^z$ is a multivalued function.

(ii) $\log z = \log r + i(\theta + 2n\pi)$ is a multivalued function.

(iii) $\operatorname{Log} z = \log r + i\alpha$; where $-\pi < \alpha \le \pi$

$\Rightarrow \log z_1 . z_2 \ne \log z_1 + \log z_2$ (In general)

$\operatorname{Log} z_1 . z_2 = \operatorname{Log} z_1 + \operatorname{Log} z_2 \pmod{2\pi}$
$= \log r_1 + \log r_2 + i[2\pi - (\alpha_1 + \alpha_2)]$

Functions :-

(i) Polynomial

(ii) Trigonometric

(iii) Hyperbolic.

$\sinh z = \dfrac{e^{z}-e^{-z}}{2}$

$\cosh z = \dfrac{e^{z}+e^{-z}}{2}$

$\sin x = \dfrac{e^{ix}-e^{-ix}}{2i}$

$\sin(ix) = \dfrac{e^{-x}-e^{x}}{2i}$

$i\sin(ix) = \dfrac{e^{-x}-e^{x}}{2}$

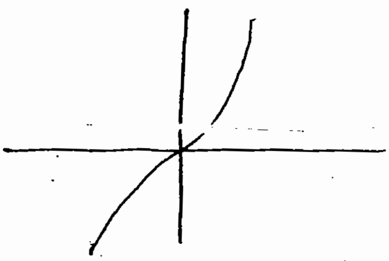
$i\sin(ix) = -\sinh x$

$\boxed{\sin(ix) = i\sinh x}$

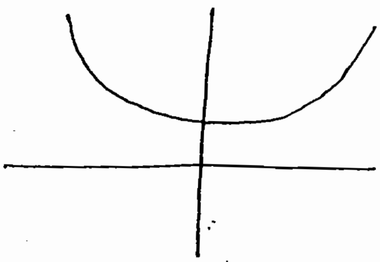
Similarly, $\boxed{\cos(ix) = \cosh x}$

$$\begin{aligned}\sin z &= \sin(x+iy)\\ &= \sin x\cdot\cos iy + \cos x\cdot\sin iy\\ &= \underset{\text{bdd}}{\sin x}\cdot\underset{\text{unbdd}}{\cosh y} + i\underset{\text{bdd}}{\cos x}\,\underset{\text{unbdd}}{\sinh y}\end{aligned}$$

sinh x



cosh x



→ sin x, cos x are periodic functions.

→ e^z is periodic function

$$f(z) = e^z$$

$$f(z+2n\pi i) = e^{z+2n\pi i}$$

$$= e^z \cdot e^{2n\pi i} \qquad [\because e^{2n\pi i} = 1]$$

$$= e^z$$

$$= f(z)$$

$$e^z = e^{x+iy} = e^x \cdot e^{iy}$$

$$|e^z| = |e^x| \cdot |e^{iy}| = e^x \times 1 \rightarrow \text{unbdd}$$

$$= e^x$$

Whereas

$$e^{iz} = e^{i(x+iy)} = e^{ix} \cdot e^{-y}$$

$$|e^{iz}| = |e^{ix}|\,|e^{-y}| = e^{-y} \text{ bdd above.}$$

e^z is also a single valued function.

$\boxed{\text{Problem}}$:- Find all solutions of $\tan z = 2$ and mention the principal value.

$\boxed{\text{Solution}}$:- The solution set is given by

$$z = \tan^{-1} 2$$

$$\tan^{-1} 2 = \frac{i}{2} \log\left(\frac{i+2}{i-2}\right)$$

$$= \frac{i}{2} \log\left\{\left(\frac{3+4i}{5}\right)\right\}$$

$$= \frac{i}{2}\left[i\left(2n\pi - \pi + \tan^{-1}\frac{4}{3}\right)\right]$$

$$= \frac{1}{2}\left[(2n+1)\pi - \tan^{-1}\frac{4}{3}\right];\ n \in \mathbb{Z}$$

Where $\operatorname{Arg}\left[-\frac{3}{5} - \frac{4}{5}i\right] = \tan^{-1}\left(\frac{4}{3}\right) - \pi$ is a unique number.

$\boxed{\text{Problem}}$:- $\sqrt{z^2+2z-1}$. Find branch point.

$\boxed{\text{Solution}}$:- $z^2+2z-1=0$

$$\Rightarrow z = \frac{-2 \pm \sqrt{4+4}}{2} = -1 \pm \sqrt{2}$$

$$z = -1+\sqrt{2},\ -1-\sqrt{2}$$

i.e Branch points $\to (-1+\sqrt{2}, 0)$ & $(-1-\sqrt{2}, 0)$

Branch cut :

$$y-0 = m(x+1-\sqrt{2})$$

and $y-0 = m(x+1+\sqrt{2})$

$\boxed{\text{Note}}$:- Using branch cut we make function single valued.

$\to$ $f(z) = \sqrt{z+2+3i} = \sqrt{(x+2)+i(y+3)}$

$(-2,-3)$ is the B.P.

& the line $(x+2)m = y+3$ is the branch line.

$\sqrt{(x-1)+iy}$

$\to$ $f(z) = \sqrt{z-1} \Rightarrow (1,0)$ is the B.P.

& the line $(x-0)m = y-1$ is the branch line

Functions of complex Variables :-

Limit :- We say $\lim_{z \to z_0} f(z) = l$ if for any $\epsilon > 0$, $\exists \delta > 0$ s.t. $|f(z) - l| < \epsilon$ whenever $0 < |z - z_0| < \delta$

We say $f(z)$ approaches to 'l' or lim of $f(z)$ at $z = z_0$ is l if $f(z)$ approaches to 'l', whenever z approaches to z_0 in any manner.

OR,

$$w = f(z) = u + iv$$

$$\lim_{z \to z_0} f(z) = \lim_{(x,y) \to (x_0,y_0)} u(x,y) + \lim_{(x,y) \to (x_0,y_0)} v(x,y)\, i$$

$$= l_1 + l_2 i$$

where $l_1 + l_2 i = l$.

$\rightarrow$ $\lim_{z \to z_0} f(z)$ exists iff $\lim_{(x,y) \to (x_0,y_0)} u(x,y)$ & $\lim_{(x,y) \to (x_0,y_0)} v(x,y)$ exi

e.g. (i) $f(z) = \dfrac{\bar{z}^2}{z}$

$$= \frac{(x - iy)^2}{x + iy} \times \frac{x - iy}{x - iy}$$

$$= \frac{(x - iy)^3}{x^2 + y^2}$$

$$= \frac{x^3 - (iy)^3 - 3xyi\,(x - iy)}{x^2 + y^2}$$

$$u + iv = \frac{x^3 - 3xy^2}{x^2 + y^2} - i\,\frac{(3x^2y - y^3)}{x^2 + y^2}$$

$$u = \frac{x^3 - 3xy^2}{x^2 + y^2}, \quad v = -\frac{(3x^2y - y^3)}{x^2 + y^2}$$

$$\left|\frac{x^3-3xy^2}{x^2+y^2}-0\right| \le \left|\frac{x^3}{x^2+y^2}\right| + \left|\frac{3xy^2}{x^2+y^2}\right|$$

$$= |x|\left|\frac{x^2}{x^2+y^2}\right| + 3|x|\left|\frac{y^2}{x^2+y^2}\right|$$

$$\le |x| + 3|x|$$

$$= 4|x|$$

$$= 4|x-0| + 0.|y-0|$$

$$= 4\delta$$

$\lim\limits_{(x,y)\to(0,0)} u(x,y)$ exists

Similarly $\lim\limits_{(x,y)\to(0,0)} v(x,y)$ exists.

Hence $\lim\limits_{(x,y)\to(0,0)} f(z)$ exists.

$\boxed{\text{e.g}}$: (ii) $\lim\limits_{(x,y)\to(0,0)} \frac{\bar{z}}{z}$ does not exist or $\lim\limits_{z\to 0} \frac{\bar{z}}{z}$ does not exist.

$\boxed{\text{Solution}}$:- $\lim\limits_{(x,y)\to(0,0)} \frac{x-iy}{x+iy}$

Put $y=mx$

$$\lim_{(x,y)\to(0,0)} \frac{\cancel{x}(1-im)}{\cancel{x}(1+im)} = \frac{1-im}{1+im}$$

This is dependent on m, Hence limit does not exist.

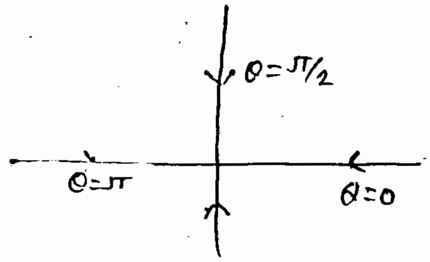
$$\rightarrow \quad \lim_{z\to 0}\frac{\bar{z}}{z} = \lim_{(x,y)\to(0,0)} \frac{x-iy}{x+iy} = \begin{cases} 1 & \text{along } x\text{-axis} \\ -1 & \text{along } y\text{-axis.} \end{cases}$$

Hence limit does not exist.

V. imp

e.g. (iii) $f(z) = e^{-1/z^2}$. Show that at $\underline{z=0}$ limit does not exis

Solution :- Path ①



$\boxed{\gamma_1 : \theta = 0}$

$$\lim_{z \to 0} e^{-1/z^2} = \lim_{r \to 0} e^{-\frac{1}{r^2 e^{2i0}}}$$

$$= \lim_{r \to 0} e^{-1/r^2} = 0$$

$\boxed{\gamma_2 : \theta = \pi/2}$

$$\lim_{z \to 0} e^{-1/z^2} = \lim_{r \to 0} e^{-\frac{1}{r^2 e^{i\pi}}} = \lim_{r \to 0} e^{\frac{1}{r^2}} = \infty$$

→ Infact $\boxed{f(z) = e^{-\frac{1}{z^m}} \quad ; m \in \mathbb{N}}$

limit does not exist at $z=0$

$\gamma_1 : \theta = 0$, $\gamma_2 : \theta = \pi/2$

Note :- We check limit of only single valued function and at the point which are member of the domain of definition or at the limit point of domain. Moreover, if $f(z)$ is a multivalued, we first make to a single valued by branch cut and then find limit at those points which are the limit point of domain of definition but not the branch point. The points on branch cut may have limit or we can discuss the limit at the points on the branch cut.

→ $f(z) = \log z$. Show that $\lim\limits_{z \to 0} f(z)$ does not exist for any $a \in (-\infty, 0)$.

→ $f(z) = \dfrac{\sin\sqrt{z}}{\sqrt{z}}$ (is a single valued function). Limit exist

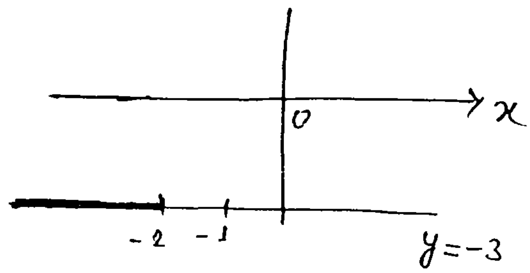
$z = re^{i\theta}$, $f(z) = \dfrac{\sin\sqrt{r}\, e^{i\theta/2}}{\sqrt{r}\, e^{i\theta/2}}$

→ $f(z) = \log(z+2+3i)$. Find the set of points where limit does not exist.

$\boxed{\text{Sol}^n}$:- $\log(z+2+3i) = \log[(x+2)+i(y+3)]$

$S = \{(x,-3) : x \in (-\infty,-2]\}$

Here $(-2,-3)$ is the B.P.



→ $\log(f(z)) = \log(u+iv)$

Limit does not exist at those points where u is $-ve$ & $v=0$.

⟨Some Results on Limit⟩

Algebra of limit

If $\lim_{z \to z_0} f(z) = l$ and $\lim_{z \to z_0} g(z) = m$ then

(i) $\lim_{z \to z_0} [f(z) \pm g(z)] = l \pm m$

(ii) $\lim_{z \to z_0} c\, f(z) = cl$; where 'c' is any complex number.

(iii) $\lim_{z \to z_0} f(z) \cdot g(z) = lm$

(iv) $\lim_{z \to z_0} \frac{f(z)}{g(z)} = \frac{l}{m}$; provided $m \neq 0$.

$\boxed{\text{Continuity}}$:- $W = f(z)$ is continuous at $z = z_0$ if for any $\varepsilon > 0$, $\exists\ \delta > 0$

s.t. $|f(z) - f(z_0)| < \varepsilon$ whenever $|z - z_0| < \delta$

i.e. We say $W = f(z)$ is continuous at $z = z_0$ if it is defined there at and $\lim_{z \to z_0} f(z) = f(z_0)$.

$\boxed{\text{Note}}$:- (i). At isolated points of the domain, functions are assumed to be continuous.

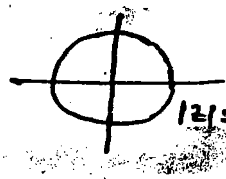
(ii) At the limit points of the domain, the function is discontinuous if they are not the members of the domain

$\boxed{\text{e.g.}}$:- (i) $f(z) = \sin z$; $z \neq 1$

$\Rightarrow$ f is discontinuous at $z = 1$

(ii) $f(z) = \sin z$; $z \in D : |z| \leq 1 = S \cup \{2 + 3i\}$

Hence $f(z)$ is continuous on $|z| \leq 1$ & $z = 2 + 3i$.



$|z| = 1$

Removable discontinuity: If $\lim_{z \to z_0} f(z)$ exists but either not equal to $f(z_0)$

Or, $f(z)$ is not defined at z_0 then the function $f(z)$ have removable discontinuity at z_0.

e.g. (i) $f(z) = \frac{\sin z}{z}$; $z \neq 0$

(ii) $f(z) = \sin z$; $z \neq 0$

(iii) $f(z) = \begin{cases} \frac{\sin z}{z} & ; z \neq 0 \\ 0 & ; z = 0 \end{cases}$

Some results on continuity:-

1). If $f(z)$ and $g(z)$ are continuous functions at z_0 then

(i) $f(z) \pm g(z)$ is also continuous.

(ii) $f(z) \cdot g(z)$ is also continuous.

(iii) $f(z)/g(z)$ provided $g(z) \neq 0$ is also continuous.

2). A polynomial is continuous at each point of a complex plane.

3). If a function $f(z)$ is continuous at z_0 and $f(z_0) \neq 0$ then $\exists$ a nbd of z_0 in which $f(z) \neq 0$.

4). If a function is continuous on a bounded set $D \subset \mathbb{C}$ then min^m and max^m of $|f(z)|$ exist on D.

Uniform continuity in complex plane:- A function $w = f(z)$ is said to be uniformly continuous on a set $D \subset \mathbb{C}$ if for given $\varepsilon > 0$ $\exists$ a $\delta > 0$ s.t.

$$|f(z_1) - f(z_2)| < \varepsilon \text{ whenever } |z_1 - z_2| < \delta \quad \forall z_1, z_2 \in D.$$

Result:– Let $f(z)$ be a continuous function on a closed & bounded set $D \subset \mathbb{C}$ then $f(z)$ is uniformly continuous on D.

Differentiability:– A function $w = f(z)$ is said to be differentiable at z_0 if

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = f'(z_0) \text{ exists finitely and this limit}$$

is denoted by $f'(z_0)$ and called derivative of $f(z)$ at z_0

OR,

$$\lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = f'(z_0).$$

Another definition: $\frac{f(z) - f(z_0)}{z - z_0} - f'(z_0) = \phi(z)$

Where for any $\epsilon > 0$, $\exists\ \delta > 0$ s.t.

$$|\phi(z)| < \epsilon \quad \text{whenever } |z - z_0| < \delta$$

$$\Rightarrow \boxed{f(z) = f(z_0) + (z - z_0) f'(z_0) + (z - z_0)\,\phi(z)}$$

Result: If $f(z)$ and $g(z)$ are differentiable at $z = z_0$ then

(i) $\frac{d}{dz}[f(z) \pm g(z)] = f'(z) \pm g'(z)$

(ii) $\frac{d}{dz}[c \cdot f(z)] = c f'(z)$

(iii) $\frac{d}{dz}[f(z) \cdot g(z)] = f'(z) \cdot g(z) + g'(z) f(z)$

(iv) $\frac{d}{dz}[f(z)/g(z)] = \frac{f'(z) g(z) - f(z) g'(z)}{[g(z)]^2}$

provided $g(z) \neq 0$.

Note: In limit, continuity & differentiability, limit should exist along any direction and should be equal in any direction.

Neighbourhood:- For any $\delta > 0$, $|z-z_0| < \delta$ is considered as a δ-nbd of z_0.

Interior point of a set:- Let $D \subseteq \mathbb{C}$ and z_0 be a member of D. We say z_0 is an interior point of D if $\exists$ a nbd of z_0 which is contained in D.

i.e. $\exists\ \delta$ s.t. $|z-z_0| < \delta \subset D$.

Boundary Point:- Let $z_0 \in \mathbb{C}$, it is said to be a boundary point of D if for any δ, $|z-z_0| < \delta \cap D \neq \phi$ as well as $|z-z_0| < \delta \cap D^c \neq \phi$ and denote this $\underline{\partial D}$ and called boundry of D.

Exterior Point:- A point of the complex plane, which is neither interior nor boundry point is called an exterior point.

Open set:- A set is said to be open if every point of the set is an interior point <u>or</u>. A set that does not have any of its boundry point but interiors is called an open set.

Closed set:- A set which is not open is called closed set.

A closed set contains all its boundry points.

Note (i) The intersection of finite numbers of open sets is open.

(ii) The arbitrary union of open sets is open.

Function of two variables :-

Limit :- We say $\lim\limits_{(x,y)\to(a,b)} u(x,y) = l$

where $u : \mathbb{R}^2 \to \mathbb{R}$

if for any $\varepsilon > 0$, $\exists\ \delta > 0$

s.t. $|u(x,y) - l| < \varepsilon$

$$0 < |x-a| < \delta$$
$$0 < |y-b| < \delta$$

Algorithm :- Simplify to get

$$|u(x,y) - l| \le \alpha|x-a| + \beta|y-b|$$
$$\le \delta(\alpha+\beta) = \varepsilon \Rightarrow \delta = \frac{\varepsilon}{\alpha+\beta}$$
$$< \varepsilon$$

$$0 < |x-a| < \delta = \frac{\varepsilon}{\alpha+\beta}$$

$$0 < |y-b| < \delta = \frac{\varepsilon}{\alpha+\beta}$$

OR,

$$|u(x,y) - l| \le \alpha|x-a||y-b|$$
$$< \alpha\delta^2 = \varepsilon \qquad \delta = \sqrt{\varepsilon/\alpha}$$
$$< \varepsilon$$

$$0 < |x-a| < \delta = \sqrt{\varepsilon/\alpha}$$

$$0 < |y-b| < \delta = \sqrt{\varepsilon/\alpha}$$

Problem :- To show $\lim\limits_{(x,y)\to(2,3)} (4x+3y) = 17$

Solution :- Now, $|4x+3y-17| = |4(x-2)+3(y-3)|$

$$\le 4|x-2| + 3|y-3|$$
$$\le \delta(4+3)$$
$$< 7\delta = \varepsilon \qquad \delta = \varepsilon/7$$
$$< \varepsilon$$

$0 < |x-2| < \delta = \varepsilon/7$, $0 < |y-3| < \delta = \varepsilon/7$

Problem:- To show $\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}} = 0$

Solution:- $\left|\frac{xy}{\sqrt{x^2+y^2}} - 0\right| = \frac{|x|\,|y|}{\sqrt{x^2+y^2}}$

$\le |x|$

$\le 1.|x-0| + 0.|y-0|$

$< \delta(1+0)$

$< \delta = \varepsilon$

$y^2 \le x^2+y^2$

$|y| \le \sqrt{x^2+y^2}$

$\frac{|y|}{\sqrt{x^2+y^2}} \le 1$

$0 < |x-0| < \delta = \varepsilon$

$0 < |y-0| < \delta = \varepsilon$

Problem:- $u(x,y) = x\sin\frac{1}{y} + y\sin\frac{1}{x}$

Show that $\lim_{(x,y)\to(0,0)} u(x,y) = 0$

Solution:- $\left|x\sin\frac{1}{y} + y\sin\frac{1}{x}\right| \le |x|\left|\sin\frac{1}{y}\right| + |y|\left|\sin\frac{1}{x}\right|$

$\le |x| + |y|$ $\quad \boxed{|\sin x| \le 1}$

$\le 1.|x-0| + 1.|y-0|$

Now, $\left|x\sin\frac{1}{y} + y\sin\frac{1}{x} - 0\right| < \delta(1+1)$

$< 2\delta = \varepsilon \quad ; \delta = \varepsilon/2$

$0 < |x-0| < \delta = \varepsilon/2$

$0 < |y-0| < \delta = \varepsilon/2$

Note:-

→ Limit, continuity & differentiability are the properties of the function at a point which even can be isolated.

To show a function of two variables is discontinuous:

We should show either function is not defined at (a,b) or limit does not exist at this point or limit exist but not equal to the define value.

To show this specially the failure of existence of limit. We should try to find two different path along which we get the different lim.

e.g.: $u(x,y) = \frac{x^2-y^2}{x^2+y^2}$, $\lim_{(x,y)\to(0,0)} u(x,y)$ does not exist

Solution:- $\lim_{(x,y)\to(0,0)} u(x,y) = \lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$

put $y = mx$

$= \lim_{x\to 0} \frac{1-m^2}{1+m^2} \rightarrow$ which depends on m.

Hence limit does not exist.

V.imp

Problem:- $f(z) = e^{-1/z^2}$

$\lim_{z\to 0} f(z) = ?$

Solution:- $z = re^{i\theta}$, where $r\to 0$, $z\to 0$

$$\lim_{z\to 0} e^{-1/z^2} = \lim_{r\to 0} e^{-\frac{1}{r^2 e^{2i\theta}}}$$

Case I:- when $\theta = 0$ i.e. moving along x-axis

$$\lim_{r\to 0} e^{-1/r^2} = 0$$

Case II:- when $\theta = \pi/2$ i.e. moving along y-axis

$$\lim_{r\to 0} e^{-\frac{1}{r^2 e^{i\pi}}} = \lim_{r\to 0} e^{1/r^2} = \infty$$

Hence $\lim_{z\to 0} f(z)$ does not exist.

V. Imp
Result

$f(z) = e^{-1/z^m}$

$\lim_{z \to 0} e^{-1/z^m}$ does not exist $\forall\, m \in \mathbb{N}$

Regular Point :- Let $w = f(z)$ be defined on D and we say z_0 is a regular point of $f(z)$ if $f(z)$ is differentiable at every point of the disc $|z - z_0| < \delta$

i.e. A point complex number z_0 is said to be a regular point of $f(z)$ if $\exists$ a δ nbd of z_0 in which $f(z)$ is everywhere differentiable.

Analiticity / Analytic function :- Let $f(z)$ be defined on D. We say $f(z)$ is analytic in D if every point of D is regular point of $f(z)$. Hence automatically implies D has to be open.

OR, Let D be an open region (domain) in $\mathbb{C}$. We say $f(z)$ is analytic in D if it is differentiable at every point of D.

Note :- If it is referred that $f(z)$ is analytic at $z = z_0$, it implies $\exists$ a δ-nbd of z_0 i.e. $D : |z - z_0| < \delta$ s.t. $f(z)$ is differentiable at every point of D.

→ If $f(z)$ is differentiable at finite points or countable points $\Rightarrow$ $f(z)$ is not analytic because $f(z)$ is analytic at $z = a$ iff $f(z)$ is analytic at disc (open) and disc has uncountable number of points.

Singularity :- (Extra Ordinary Behaviour)

If a single valued function $f(z)$ fails to be analytic at $z=z_0$ but analytic in some where in domain D containing z_0. Then $z=z_0$ is called singularity of $f(z)$.

[OR],

If $f(z)$ fails to be differentiable at z_0 and z_0 is the limit point of regular point of $f(z)$ then we say z_0 is the singular point of $f(z)$ and $f(z)$ has the singularity at $z=z_0$.

i.e. If z_0 is singularity of $f(z)$ then in every nbd of z_0 there are points s.t. $f(z)$ is regular or analytic in some nbd of these points.

Note (i) If $f(z)$ is nowhere analytic in D then it has no singularity.

E.g.: (a) $f(z)=|z|$. It is nowhere analytic
$\Rightarrow$ It has no singularity.

(b) $f(z)=|z|^2$. It is nowhere analytic
$\Rightarrow$ It has no singularity.

(c) $f(z)=\sin\bar{z}$. It is nowhere analytic
$\Rightarrow$ It has no singularity.

(d) $f(z)=e^{\bar{z}}$. It is nowhere analytic
$\Rightarrow$ It has no singularity.

(e) $f(z)=e^{-1/z^m}$; $m\in\mathbb{N}$
It is nowhere analytic
$\Rightarrow$ It has no singularity

(f) $f(z)=\bar{z}$
It is nowhere analytic
$\Rightarrow$ It has no singularity

→ (a) $f(z)=|z|=\sqrt{|z|^2}=\sqrt{z\cdot\bar{z}}$

$\frac{\partial f}{\partial \bar{z}}=\frac{1}{2}\sqrt{\frac{z}{\bar{z}}}\neq 0$. So not diff. at any pt. So nowhere analytic (limit does not exist at $z=0$)
i.e. this function has no singularity.

(b) $f(z) = |z|^2 = z \cdot \bar{z}$

$\frac{\partial f}{\partial \bar{z}} = z$, $\frac{\partial f}{\partial \bar{z}} = 0 \Rightarrow z = 0$

This function has no singularity.

(c) $f(z) = \sin \bar{z}$

$\frac{\partial f}{\partial \bar{z}} = \cos \bar{z} = 0$

$\bar{z} = (2n+1)\frac{\pi}{2}$

function is differentiable only at these pts. and nowhere differentiable but nbd of odd multiple of $\pi/2$ function is not analytic.

Hence function has no singularity.

$\boxed{\text{Entire Function}}$:- A complex function $f(z)$ is said to be entire if it is analytic in the whole complex plane.

→ The sum, difference and the product of two or more entire functions are also entire function. The ratio of entire functions is also entire function provided the denominator is non-zero. Moreover, the superposition of an entire function is again an entire function. $\boxed{\text{Superposition}}$ $g \circ f$ → composition of f with g $\boxed{\text{or}}$, superposition of g on f.

→ The following functions are entire functions :-

(i) All polynomials

i.e. $P(z) = \sum_{i=0}^{n} a_i z^i$; $a_i \in \mathbb{C}$

(ii) $\sin z$ / $\cos z$

(iii) e^z

(iv) $\log z$; $z \in \mathbb{C} - (-\infty, 0]$ → This function is analytic but not entire in $\mathbb{C}$

Classification of Singularity :-

1). Character based classification.

2). Position based classification.

Character based classification :-

1). **Branch point** :- Branch point is the isolated singularity

2). **Removable singularity** :- If $\lim_{z \to z_0} f(z)$ exists but not equal to $f(z_0)$ or $f(z)$ is not defined at $z = z_0$ then z_0 is said to be removable singularity.

OR, If $f(z)$ is analytic in $D - \{z_0\}$ or $D/\{z_0\}$ and $\lim_{z \to z_0} f(z)$ exists finitely, then z_0 is said to be removable singularity. (Here z_0 is the l.p. of D)

e.g (i) $f(z) = \sin z \quad ; z \neq 0$

(ii) $f(z) = \dfrac{\sin z}{z} \quad ; \quad z \neq 0$

(iii) $f(z) = \begin{cases} \dfrac{\sin z}{z} & ; \quad z \neq 0 \\ 0 & ; \quad z = 0 \end{cases}$

(iv) $f(z) = e^{z} \quad ; \quad z \neq 0$

Here $z = 0$ is the removable singularity.

Note V.imp (i) In some nbd of removable singularity $f(z)$ is bdd

(ii) $f(z)$ can be made analytic at removable singularity by redefining the function or extending the function at that point.

(iii) Some times in the analysis of a singular point for the function, removable singularity is not even consider for the singular point.

→ Extension of functions mean extension of domain of defⁿ.

3). **Pole** :- If $\lim_{z\to z_0} f(z)$ does not exist but $\exists\ m \in \mathbb{N}$

s.t. $\lim_{z\to z_0} (z-z_0)^m f(z)$ exists finitely and non-zero

then we say z_0 is the pole of order m.

If $m=1$, then it is a simple pole.

e.g. (i) $f(z) = \frac{\sin z}{z^5}$

$$\lim_{z\to 0} (z-0)^4 \frac{\sin z}{z^5} = 1 \neq 0$$

$\Rightarrow z=0$ is pole of order 4.

(ii) $f(z) = \frac{1}{z-a}$

$\lim_{z\to a} \frac{1}{z-a}$ does not exist but $\lim_{z\to a} (z-a)\frac{1}{(z-a)} = 1 \neq 0$

Here $z=a$ is the simple pole.

(iii) $f(z) = \frac{\cos z}{(z-1)^2}$; $m=2$ ($z=1$ is the pole of order 2)

(iv) $f(z) = \frac{\cos z}{z}$; $m=1$ ($z=0$ is the simple pole)

V.Imp → $\frac{\sin z}{z^n}$, $z=0$ is the pole of order $(n-1)$

V.Imp → $\frac{\cos z}{z^n}$; $z=0$ is the pole of order n.

V.Imp → $\frac{\tan z}{z^n}$; $z=0$ pole of order $n-1$

V.Imp → $\frac{\cot z}{z^n}$; $z=0$ pole of order $n+1$

4). $\boxed{\text{Essential Singularity}}$:- Singularity which is neither removable nor pole is defined as essential singularity. In this case limit does not exist.

$\boxed{e.g.}$ (i) $f(z) = e^{1/z^2}$; at $z=0$ is E.S.

(ii) $f(z) = \sin\frac{1}{z}$; at $z=0$ is E.S.

(iii) $f(z) = z^2 \sin\frac{1}{z}$; at $z=0$ is E.S.

Isolated.

(iv) $f(z) = \tan\frac{1}{z}$; at $z=0$ is E.S. (non-isolated)

(v) $f(z) = \log z$ has only essential singularity at all $-ve$ points except $z=0$ (B.P)

(vi) $f(z) = e^{1/z}$; at $z=0$ is E.S. (isolated)

(vii) $f(z) = \cot\frac{1}{z}$ at $z=0$ is E.S. (Non-isolated)

V.Imp

$\boxed{e.g.}$: $f(z) = \begin{cases} e^{-\frac{1}{z^m}} & ; z \neq 0 \\ 0 & ; z=0 \end{cases}$ $m \in \mathbb{N}$

$$\lim_{z\to 0} e^{-1/z^m} = \lim_{r\to 0} e^{-1/r^m} e^{-im\theta} = \begin{cases} 0 & r\to 0 \text{ on } \theta=0 \\ \infty & r\to 0 \text{ on } \theta=\pi/ \end{cases}$$

$\Rightarrow \lim_{z\to 0} f(z)$ does not exist.

V.Imp

$\rightarrow$ If $\lim_{z\to 0} z^t e^{-1/z^m}$ does not exist for any t.

Hence $z=0$ is essential singularity.

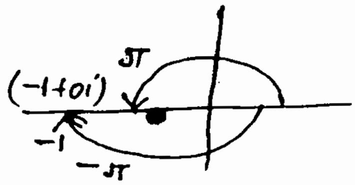
→ $\sec\frac{1}{z}$; at $z=0$ is E.S (Non-isolated)

→ $\operatorname{cosec}\frac{1}{z}$; at $z=0$ is E.S. (Non-isolated)

V. imp

Problem:- Find the singularity of $\log z$ in domain $D : \mathbb{C} - \{0\}$

Solution:- $w = \log z$; $z \in \mathbb{C} - \{0\}$



$z = 0$ is branch point of $\log z$.

→ $w = \log z$ is analytic at $\mathbb{C} - (-\infty, 0]$

$\log z = \log(-1 + 0.i) = \log 1 + i \tan^{-1}\frac{0}{-1}$
$= 0 + i \cdot 0$

$\lim\limits_{z \to -1} \log z$ does not exist as along any path in IInd quadrant, the ~~imaginary~~ argument approaches π as $z \to -1$ where as along any path in IIIrd quadrant, the imaginary of $\log z$ approaches $-\pi$ as $z \to -1$.

Hence $\lim\limits_{z \to a} \log z$ does not exist $\forall\, a \in (-\infty, 0)$

Also $\lim\limits_{z \to a} (z-a)^r \log z = 0$, so $\log z$ always have essential singularity.

Hence at every negative real no. $\log z$ has ~~an~~ essential singularity.

E.g.:- $w = \log(z-2)$

$z = 2$ is branch point

w is analytic at $\mathbb{C} - (-\infty, 2]$.

V. imp. → If $\lim\limits_{z \to a} f(z)$ does not exist, then $z = a$ is an essential singularity.

imp → Limit point of zeros is an isolated essential singularity.

V. imp → Limit point of poles is a non-isolated essential singularity.

Algebra of analytic functions :-

1). If f and g are analytic functions, then $f+g$, $f-g$, $af+bg$, $f\cdot g$ are analytic in the common domain and f/g ($g\neq 0$) is analytic in the common domain except at $g\neq 0$.

2). $\sin z$, $\cos z$, all polynomials are analytic in $\mathbb{C}$.

V.Imp 3). If f and g are differentiable at z_0 then fog or gof whichever is defined is differentiable at z_0.

Problem :- $f(z) = \sin\frac{1}{(z-a)}$, Find limit point of zeros; (limit point of zeros is an isolated essential singularity)

Solution :- Zeros of $f(z)$ is obtained by putting

$$f(z)=0$$

$$\sin\left(\frac{1}{z-a}\right)=0 \Rightarrow \frac{1}{z-a}=n\pi$$

$$\text{or, } z = a+\frac{1}{n\pi} \quad ; \; n\in\mathbb{Z}$$

Hence limit point is 'a'

Also $z=a$ is an isolated essential singularity of the function $\sin\left(\frac{1}{z-a}\right)$.

Problem :- $f(z) = \dfrac{1}{\sin\left(\frac{1}{z-a}\right)}$

Find limit point of poles (i.e. limit point of poles is a non-isolated essential singularity)

Solution :- Poles of $f(z)$ is obtained by putting denominator equal to zero

i.e $\sin\left(\frac{1}{z-a}\right)=0 \Rightarrow \sin\left(\frac{1}{z-a}\right) = \sin n\pi \Rightarrow \frac{1}{z-a}=n\pi$

$\Rightarrow z = a + \frac{1}{n\pi}$

Hence limit point of poles is $z=a$.

Also $z=a$ is non-isolated essential singularity.

Algorithm for Singularities :-

Let $w = f(z)$ be given

Step-I :- Test for single or multivalued complex function.

If single valued function then

Step-II :- Collect the points in the domain or testing point where function is either not define or takes the indeterminate form and try to find limit at these points. If exist claim for removable singularities. If do not exist then go to step-III.

Note :- For step-II one should take help of class of analytic function or algebra of analytic function for collecting point

e.g. :- $f(z) = \frac{\sin z}{z+2z^2}$

$\sin z$ is analytic and $z+2z^2$ is also analytic so $f(z)$ is also analytic provided denominator $\neq 0$.

Step-III :- If at z_0 limit does not exist (∞)

Construct $g(z) = (z-z_0)^m f(z)$; $m \in \mathbb{N}$

and then try to find $\lim_{z \to z_0} g(z)$ = exists finitely and non-zero.

If 'm' is obtained, then claim for z_0 is pole of order 'm', if no such 'm' exists then claim for z_0 has an essential ~~singularity~~.

Note : 1 :- In case of deciding the nature of $f(z)$ at infinit construct $g(z) = f\left(\frac{1}{z}\right)$
then nature of $g(z)$ at origin is same as nature of $f(z)$ at ∞, $z=\infty$ (special)

Note 2 :- If variable (z) is in exponent of variable or the exponent of variable is not integer [e.g. (i) z^z (ii) z^i] then one should use the concept $f(z) = e^{\log(f(z))}$ and then solve.

V.Imp.

Problem :- $f(z) = z^z$

Solution :- $f(z) = e^{\log z^z} = e^{z\log z}$ → multivalued

$z=0$ is B.P.

$f(z) = \dfrac{1}{\sin\left(\frac{1}{z}\right)}$

Sol^n :- Put $\sin\left(\frac{1}{z}\right)=0 \Rightarrow \frac{1}{z} = n\pi \Rightarrow z = \frac{1}{n\pi}$

$$\lim_{z \to \frac{1}{n\pi}} \frac{1}{\sin\left(\frac{1}{z}\right)}$$ does not exist

Now. $g(z) = \left(\frac{1}{z} - n\pi\right) \cdot \dfrac{1}{\sin\left(\frac{1}{z}\right)}$

$$\lim_{z\to 1/n\pi} g(z) = \lim_{z \to 1/n\pi} \frac{\left(\frac{1}{z} - n\pi\right)}{\sin(1/z)} \quad \left[\frac{0}{0} \text{ forom}\right]$$

$$= \lim_{z\to 1/n\pi} \frac{-1/\cancel{z^2}}{\cos(1/z) \times \cancel{(-1/z^2)}} = \lim_{z\to 1/n\pi} \sec 1/z = \sec n\pi$$

exists finite and 1 or -1

Hence $1/n\pi$ is pole of order one (infinite poles)

i.e. $z = \frac{1}{n\pi}$ is simple pole. Limit point $z=0$ non-isolated essentia

Positional classification of singularities

1) **Isolated Singularity**:- Let $z=z_0$ be the singularity of $f(z)$. We say $z=z_0$ is an isolated singularity of $f(z)$ if $\exists$ a δ nbd. of $z=z_0$ in which there is no singularity of $f(z)$ except $z=z_0$.

i.e. If z_0 is a singularity of $f(z)$ such that $f(z)$ is analytic in the deleted nbd of z_0.

i.e. Singularity can be isolated from the other singularity.

i.e If we can find a nbd in which there is no other singularity.

e.g:- $f(z) = \frac{1}{z-1}$; $z=1$ is isolated singularity.

2). **Non-Isolated Singularity**:- A singularity is said to be non-isolated if every nbd of that singularity contains at least one singularity other than that singularity. Hence we can say $z=z_0$ is a non-isolated singularity if every nbd of z_0 contains infinite singularities of $f(z)$:

i.e. Let z_0 be a singularity, we say it is non-isolated singularity if for any δ, there is at least one singularity in the deleted nbd of z_0 namely $0<|z-z_0|<\delta$

i.e. If a function has finite number of singularities, then all singularities are isolated.

i.e. for a function to have non-isolated singularities it is necessary it has infinite number of singularities but not sufficient. Sufficiency comes when we take the extended complex plane.

Note:- i). Limit point of singularities is always non-isolated essential singularities.

ii). Poles are isolated by definition. Hence poles are isolated singularities.

iii). Removable singularities are also isolated singularities.

V.V.I iv). Every non-isolated singularity is an essential singular but not conversely.

e.g. $f(z) = e^{1/z}$.

v). In an analytic function, zero's of analytic function isolated provided it is not identically zero.

$$W = f(z) \text{ is non-identically}$$

$$S = \{z \in D : f(z) = 0\}$$

We say 'l' is a point of S

$$z_n = |z - l| < 1/n, \quad z_n \in S.$$

$$f(z_n) = 0$$

$$\langle f(z_n) \rangle \longrightarrow 0, \quad z_n \longrightarrow l$$

$$f(z_n) \rightarrow f(l), \quad l \text{ is zero.}$$

So limit point of zero's is an isolated essential singularities.

vi). Limit point of poles is a non-isolated essential singularity

vii) If limit point of zeros of an analytic function belong to the domain of analyticity then that function is identically zero in that domain.

[Result]:-

1). If $f(z) = \frac{p(z)}{q(z)}$ where $p(z)$ and $q(z)$ are analytic then roots of $q(z)$ are poles of $f(z)$ provided they are not the roots of $p(z)$ as well.

2). If $z = z_0$ is pole of $f(z)$ then at every nbd of $z = z_0$, $f(z)$ is unbdd.

3). If S is set of zeros of $f(z)$ then limit point of S is essential singularity of $f(z)$ or else function is identically zero.

[Problem]: ① Find the singularities and classify them.

$$f(z) = \frac{\cos 1/z}{\sin 1/z} = \cot 1/z$$

[Solution]: $\sin 1/z = 0$

$$\Rightarrow \frac{1}{z} = n\pi$$

$$\Rightarrow z = 1/n\pi$$

0 is the limit point of poles (singularity) is non-isolated essential singularity.

Also $\cos 1/z = 0 \Rightarrow 1/z = (2n+1)\pi/2$

$$\Rightarrow z = \frac{2}{(2n+1)\pi}$$

$\Rightarrow$ 0 is also the limit point of zeros of $f(z)$

$\Rightarrow$ 0 is also essential singularity.

Another method :- $z=0$, $z=1/n\pi$, $z=\infty$

At $z = 1/n\pi$

$$\lim_{z\to 1/n\pi} f(z) = \lim_{z\to 1/n\pi} \cot(1/z) = \cot n\pi \text{ (does not exist)}$$

Now, construct, $g(z) = \left(z - \frac{1}{n\pi}\right)\cot\left(\frac{1}{z}\right)$

$$\lim_{z\to \frac{1}{n\pi}} g(z) = \lim_{z\to 1/n\pi} (z - 1/n\pi)\cdot\cot(1/z)$$

$$= \lim_{z\to 1/n\pi} \frac{(z-1/n\pi)}{\tan(1/z)} \quad \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{z\to 1/n\pi} \frac{1}{\sec^2(1/z)(-1/z^2)}$$

$$= \lim_{z\to 1/n\pi} \frac{-z^2}{\sec^2(1/z)} = \frac{-1/n^2\pi^2}{1}$$

$$= -\frac{1}{n^2\pi^2} = \text{exists finitely and non-zero.}$$

Hence $z = \frac{1}{n\pi}$ is a simple pole isolated singularitie

Limit point of singularities is always non-isolated essentia singularities.

i.e. $\lim_{n\to\infty} z = \lim_{n\to\infty} \frac{1}{n\pi} = 0$

i.e. $z=0$ is non-isolated essential singularity.

Now at $\underline{z=\infty}$

$$f(z) = \cot(1/z)$$

Now, $g(z) = f(1/z) = \cot z$ at $z=0$

$$\lim_{z\to 0} g(z) = \lim_{z\to 0} \cot z = \text{does not exist.}$$

Now, construct $h(z) = (z-0)\cot z$

$$\lim_{z\to 0} h(z) = \lim_{z\to 0} (z-0)\cot z = \lim_{z\to 0} \frac{(z-0)}{\tan z} \quad \left(\tfrac{0}{0} \text{ form}\right)$$

$$= \lim_{z\to 0} \frac{1}{\sec^2 z} = 1 \neq 0$$

i.e. $z=0$ is simple pole for $h(z) = \cot z$

Therefore $\underline{z=\infty}$ $\underline{\text{is simple pole for } f(z) = \cot(1/z)}$

Conclusion:-

$z = 1/n\pi$ is a simple pole isolated singularities

$n \in I - \{0\}$ $z=0$ is a non-isolated essential singularity

$z=\infty$ is a simple pole isolated singularity.

Problem:- (2) Find the singularities and classify them:

$$f(z) = \operatorname{cosec} z$$

Solution:- $f(z) = \operatorname{cosec} z = \frac{1}{\sin z}$

Point $z=0$, $n\pi$, ∞ i.e $z = n\pi$, ∞, $n = 0, \pm 1, \pm 2, \ldots$

At $z=n\pi$

$$\lim_{z\to n\pi} f(z) = \lim_{z\to n\pi} \frac{1}{\sin z} = \text{does not exist}$$

Now, construct $g(z) = (z-n\pi)\frac{1}{\sin z}$

$$\lim_{z\to n\pi} g(z) = \lim_{z\to n\pi} \frac{(z-n\pi)}{\sin z} \quad \left(\tfrac{0}{0} \text{ form}\right)$$

$$= \lim_{z\to n\pi} \frac{1}{\cos z} = \pm 1 \neq 0$$

i.e $z = n\pi$ is a simple pole isolated singularity.

At $z=\infty$

$$f(z) = \frac{1}{\sin z}$$

$$f(1/z) = \frac{1}{\sin(1/z)}$$

Now limit point of singularities is always non-isolated essential singularity.

i.e $\lim_{z\to\infty} n\pi = \infty$ i.e. $z=\infty$ is non-isolated essential singularity.

Conclusion:-

$z = n\pi$; $n \in Z$ is simple pole isolated singularity

$z = \infty$ is a non isolated essential singularity.

Problem:- (3) $f(z) = z^z$

Solution:- $f(z) = e^{\log z^z} = e^{z \log z}$

$z=0$, $z=\infty$ both are branch points.

Problem: (4) $f(z) = \dfrac{1}{\sin z - \cos z}$

Solution:- Put $\sin z - \cos z = 0$

$\Rightarrow \tan z = 1 = \tan \pi/4$

$z = n\pi + \frac{\pi}{4}$, $z = \infty$

i.e. $z = n\pi + \frac{\pi}{4}$ is a simple pole

$z = \infty$ is a non-isolated essential singularity.

Problem: (5) $f(z) = \operatorname{cosec}\left(\frac{1}{z}\right) = \frac{1}{\sin\left(\frac{1}{z}\right)}$

Solution:- $z = 0, \ \infty, \ z = \frac{1}{n\pi}$

At $z = \infty$

$f(z) = \operatorname{cosec}(1/z)$

Now $h(z) = f(1/z) = \operatorname{cosec} z$ at $z=0$ check

$\lim_{z\to 0} h(z) =$ does not exist

Now, construct $g(z) = (z-0)\operatorname{cosec} z$

$$\lim_{z\to 0} g(z) = \lim_{z\to 0} \frac{z-0}{\sin z} \quad \left(\frac{0}{0}\right)$$

$$= \lim_{z\to 0} \frac{1}{\operatorname{cosec} z} = 1 \neq 0$$

i.e. $z=0$ is a simple pole, isolated for $\operatorname{cosec} z$

$\therefore z = \infty$ " " " " $\operatorname{cosec}(1/z)$

Conclusion:-

$z = \frac{1}{n\pi}$ is a simple pole, isolated

$z = \infty$ " " " "

$z = 0$ is a non-isolated essential singularity.

Problem:- (6) $f(z) = e^{1/z}$

Solution: $z=0$, is an essential singularity

But we know that finite number of singularities then all the singularities are isolated.

Therefore $z=0$ is an isolated essential singularity.

$$\rightarrow f(z) = \frac{1}{e^{1/z}-1}$$

Poles of $f(z)$ are given by

$$e^{1/z} - 1 = 0$$

$$\Rightarrow e^{1/z} = 1$$

$$\Rightarrow e^{1/z} = e^{2n\pi i}$$

$$\Rightarrow \frac{1}{z} = 2n\pi i$$

$$\Rightarrow z = \frac{1}{2n\pi i} \Rightarrow z = \frac{-i}{2n\pi}$$

0 is the limit point of poles is non-isolated essential singularity.

※ Poles are always isolated.

NET DEC 2007

$\boxed{\text{Problem}}$:- $i^{i^{i^{\cdots\infty}}} = A+iB$

When principal branch is taken find that

$$A^2+B^2 = e^{-B\pi}$$

$$B/A = \tan\frac{A\pi}{2}$$

$\boxed{\text{Solution}}$:- We have,

$$i^{i^{i^{\cdots\infty}}} = A+iB$$

$$\Rightarrow i^{A+iB} = A+iB$$

$$\Rightarrow e^{(A+iB)\log i} = A+iB$$

$$\Rightarrow e^{(A+iB)\frac{\pi}{2}i} = A+iB \quad \left\{ \because \log i = \log 1 + i\tan^{-1}\frac{1}{0} = 0 + i\frac{\pi}{2} \right\}$$

$$\Rightarrow e^{Ai\frac{\pi}{2} - B\frac{\pi}{2}} = A+iB$$

$$\Rightarrow e^{-B\frac{\pi}{2}} \cdot e^{A\frac{\pi}{2}i} = A+iB = re^{i\theta}$$

$$r = \sqrt{A^2+B^2} = e^{-B\pi/2} \Rightarrow \boxed{A^2+B^2 = e^{-B\pi}}$$

$$\theta = \tan^{-1}\frac{B}{A} = \frac{A\pi}{2}$$

$$\Rightarrow \tan^{-1}\frac{B}{A} = \frac{A\pi}{2}$$

$$\Rightarrow \boxed{\tan\frac{A\pi}{2} = \frac{B}{A}}$$ Hence the answer.

→ $f(z) = \log z$

$z=0$ is isolated branch point.

$\frac{x<0}{y=0}$, non-isolated essential singularity.

Note :- $\log z = \log(x+iy)$ is discontinuous when $x<0$ and $y=0$ i.e. $x \in (-\infty, 0)$.

Problem :- $\log(z+i)$ find the branch point and discuss it.

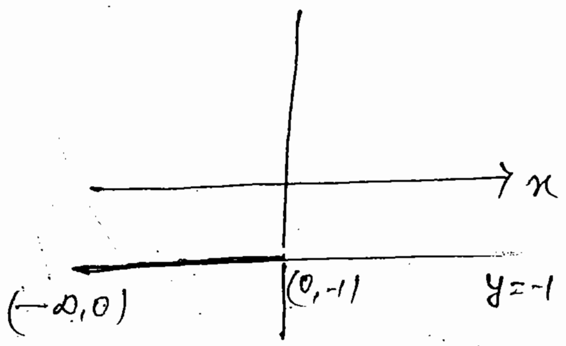
Solution :- $\log(z+i)$

$$= \log(x+iy+i)$$

$$= \log(x+i(y+1))$$

$\Rightarrow z=-i$ is branch point

The function is discontinuous on the line $y=-1$ & $x<0$



Problem:- Let $f(z) = (z^2+1)^{1/2}$. Then we find
(i) its branch points
(ii) branch line and
(iii) show that a complete circuit around these points produces no change in the branches of the function.

Solution:- (i) We have

$$w = (z^2+1)^{1/2} = [(z+i)(z-i)]^{1/2}. \text{ Then}$$

$$\arg w = \frac{1}{2}\arg(z+i) + \frac{1}{2}\arg(z-i)$$

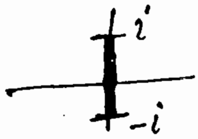
so that

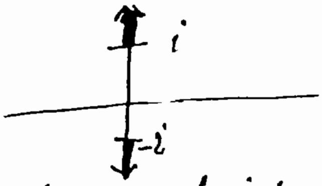
$$\text{Change in } \arg w = \frac{1}{2}[\text{Change in } \arg(z+i)] + \frac{1}{2}[\text{Change in } \arg(z-i)]$$

Let C be a circuit enclosing the point 'i' but not the point '$-i$'
The point z goes once counterclockwise around & this causes change in $\arg(z-i) = 2\pi$,
& Change in $\arg(z+i) = 0$
so that change in $\arg w = \pi$

Hence w does not return to its original value i.e. a change in branch has occured. Since the closed circuit about $z=i$ altered the branch of the function $z=i$ is a branch point.
Similarly, if C is a circuit enclosing $z=-i$, but not $z=i$ then we can show that $z=-i$ is another branch point.

(ii) The part of the y-axis $y \le -1 \, (x=0) \cup y \ge 1 \, (x=0)$ or line segment $-1 \le y \le 1 \, (x=0)$ constitutes the branch line





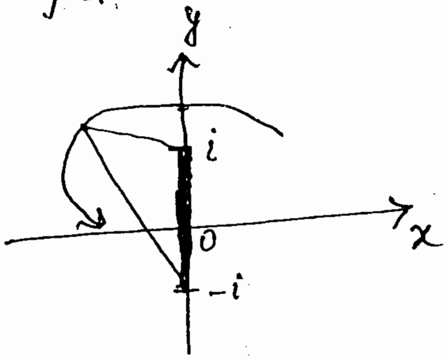
(iii) If C encloses both the branch points $z = \pm i$ and as z moves anticlockwise around C, then

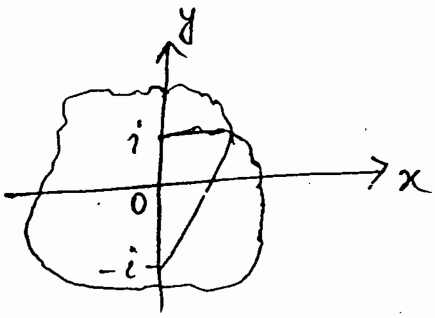
change in $\arg(z-i) = 2\pi$ &

" " $\arg(z+i) = 2\pi$

So, change in $\arg w = 2\pi$

Hence, a complete circuit around both the branch points does not change the branch.





Problem :-- Discuss $f(z) = \sin^{-1}\frac{1}{z}$ when Principal branch is taken.

Solution :- $\lim_{z \to 0} \sin^{-1}\frac{1}{z}$ = does not exist.

$\lim_{z \to 0} z^m \sin^{-1}\frac{1}{z}$, $\nexists$ no such $m \in \mathbb{N}$ so that the limit exists. Hence $z=0$ is not a pole.

Hence $z=0$ is the essential singularity.

and $1, -1$ are the branch points.

Result:- If $f(z)$ is analytic in D then at every point of D, the partial derivative of u and v exists and satisfies the equation $u_x = v_y$ & $u_y = -v_x$ called C-R equation (This is necessary condition)

Result:- If $f(z) = u+iv$ is defined on D s.t. at every point of D. u_x, u_y, v_x, v_y exists, continuous and satisfying C-R equation then $f(z)$ is analytic and

$$f'(z) = u_x + iv_x$$
$$= v_y - iu_y$$

(This is sufficient condition).

Harmonic function:-

Defn:- A function $u(x,y)$ is said to be harmonic in $D \subseteq \mathbb{R}^2$ if ~~every point of~~ u satisfies Laplacian equation at every point of D

i.e. $\nabla^2 u = 0$

i.e. $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$

Result:- If $f(z) = u+iv$ is analytic in $D \subseteq \mathbb{C}$ then at every point of D, u & v are harmonic in D and v is called harmonic conjugate of u.

Ci Theorems

C_1 :- If u and v are harmonic conjugate of each other then both are constant functions
i.e. $f(z) = u + iv$ is constant.

Proof: Since v is harmonic conjugate of u
$\Rightarrow$ $u + iv$ is analytic.
Hence, satisfy C-R equation.

$$\Rightarrow \left.\begin{aligned} u_x &= v_y \\ u_y &= -v_x \end{aligned}\right\} \text{—— (1)}$$

Also u is harmonic conjugate of v
$\Rightarrow$ $v + iu$ is analytic & satisfy C-R equation

$$\Rightarrow \left.\begin{aligned} v_x &= u_y \\ v_y &= -u_x \end{aligned}\right\} \text{—— (2)}$$

From (1) & (2), we have

$$v_x = -v_x \Rightarrow 2v_x = 0$$

$\Rightarrow$ v is function of y only

Also, $v_y = -v_y \Rightarrow 2v_y = 0$

$\Rightarrow$ v is function of x only

$\Rightarrow$ v is constant function
Similarly u is constant function.

$\boxed{C_2}$:- If $f(z) = u+iv$ is an analytic function in a domain D. If any of the following conditions are satisfied then $f(z)$ is constant.

(i) Either of u or v is constant

(ii) $|f(z)|$ is constant.

Proof :- $f(z) = u+iv$

$$|f(z)| = u^2+v^2 = k$$

$$2uu_x + 2vv_x = 0$$

$$2uu_y + 2vv_y = 0$$

$$uu_y + vv_y = 0$$

$$uu_x + vv_x = 0$$

$$-uv_x + vu_x = 0$$

$$\begin{bmatrix} u & v \\ v & -u \end{bmatrix} \begin{bmatrix} u_x \\ v_x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$AX = 0$$

$$\begin{vmatrix} u & v \\ v & -u \end{vmatrix} = -u^2-v^2 = -(u^2+v^2) = -k \neq 0$$

Thus there is only trivial solution.

$$u_x = 0$$
$$v_x = 0$$

$\Rightarrow$ u & v are constant functions

$\Rightarrow$ $f(z)$ is constant by C_2 (i).

(iii) If $\operatorname{Arg} f(z) =$ constant

Proof :- $\tan^{-1} \frac{v}{u} = k$.

$\Rightarrow v = u \tan k$

$\Rightarrow u = v \cot k$

$\Rightarrow u = cv \quad \because \cot k = c$.

We observe $u - cv = 0$ unless v is identically zero. But $u - cv$ is the real part of $f(1+ic)$, therefore it follows from C_2 (i). But $(1+ic)$ is constant, therefore f is also constant.

(iv) $u^2 = v$

(v) $v^2 = u$

Proof of (iv) :- $u^2 = v$

$$2uu_x - v_x = 0$$

& $2uu_y - v_y = 0$ or $-2uv_x - u_x = 0 \Rightarrow -u_x - 2uv_x = 0$

$$\begin{bmatrix} 2u & -1 \\ -1 & -2u \end{bmatrix} \begin{bmatrix} u_x \\ v_x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$4u^2 + 1 \neq 0$$

Thus there is only trivial solution

$\Rightarrow \left.\begin{matrix} u_x = 0 \\ v_x = 0 \end{matrix}\right\}$ are the only solutions.

$\Rightarrow u$ & v are constant function

$\Rightarrow f(z)$ is constant by C_2 (i)

(vi) $f'(z)$ vanishes identically in D.

$\boxed{C_3}$: (i) If $f(z)=u+iv$ is analytic s.t. u and v satisfying the equation

$$au+bv=c$$

where a, b, c are non-zero and $a^2+b^2\neq 0$

then $f(z)$ is constant.

$\boxed{\text{Proof}}$:- $au+bv=c$

$$au_x+bv_x=0 \Rightarrow au_x+bv_x=0$$
$$au_y+bv_y=0 \Rightarrow av_x-bu_x=0$$

$$\begin{bmatrix} a & b \\ -b & -a \end{bmatrix}\begin{bmatrix} u_x \\ v_x \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$AX=0$$

$$|A|=a^2+b^2\neq 0$$

$\Rightarrow$ only trivial solution is there

$$u_x=0$$
$$v_x=0$$

$\Rightarrow$ u & v are constant functions

Hence $f(z)=u+iv$ is constant.

$\boxed{C_3}$: (ii) If $f(z)=u+iv$ is analytic and u & v lies on a unit circle in w-plane then $f(z)$ is constant.

$\boxed{\text{Proof}}$:- $\boxed{\text{Hint}}$:- Since $u^2+v^2=1$

Further same as C_2 (ii).

Result :- If $f(z) = u + iv$ is analytic then for any α and β, $u(x,y) = \alpha$, $v(x,y) = \beta$ represents orthogonal family of curves. Product of slopes of perpendicular lines is -1 provided lines are not parallel to axis.

Solution :- If $u(x,y) = \alpha$

$$du = 0$$

$$du = u_x dx + u_y dy = 0$$

$$m_1 = \frac{dy}{dx} = -\frac{u_x}{u_y}$$

$$m_2 = \frac{dy}{dx} = -\frac{v_x}{v_y}$$

$$m_1 \cdot m_2 = \frac{u_x}{u_y} \cdot \frac{v_x}{v_y} = \frac{v_y}{-v_x} \cdot \frac{v_x}{v_y} = -1$$

$$\Rightarrow \quad m_1 \cdot m_2 = -1$$

Hence they represents orthogonal family of curves.

$\rightarrow$ $f(z) = u + iv$

$$z = x + iy$$

$$\bar{z} = x - iy$$

$$x = \frac{z + \bar{z}}{2}, \quad y = \frac{z - \bar{z}}{2i}$$

$$\frac{\partial x}{\partial \bar{z}} = \frac{1}{2}, \quad \frac{\partial x}{\partial z} = \frac{1}{2}$$

$$\frac{\partial y}{\partial \bar{z}} = -\frac{1}{2i}, \quad \frac{\partial y}{\partial z} = \frac{1}{2i}$$

$$f(z) = u + iv$$

$$f(x+iy) = u(x,y) + iv(x,y)$$

$$\frac{\partial f}{\partial \bar{z}} = \left(\frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial \bar{z}} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial \bar{z}}\right) + i\left(\frac{\partial v}{\partial x} \cdot \frac{\partial x}{\partial \bar{z}} + \frac{\partial v}{\partial y} \frac{\partial y}{\partial \bar{z}}\right)$$

$$\Rightarrow \frac{\partial f}{\partial \bar{z}} = \frac{1}{2}(u_x + i u_y) + i\frac{1}{2}(v_x + i v_y)$$

$$\Rightarrow \frac{\partial f}{\partial \bar{z}} = \frac{1}{2}(u_x - v_y) + \frac{1}{2} i (u_y + v_x)$$

If $f(z)$ is analytic

$u_x = v_y, \quad u_y = -v_x$

$$\boxed{\frac{\partial f}{\partial \bar{z}} = 0} \longrightarrow \text{complex form of C-R equation.}$$

$\Rightarrow f$ is independent of $\bar{z}$.

This is the complex form of C-R equation and completely characterises the analytic function i.e. $f(z)$ is analytic, it is independent of $\bar{z}$.

Similarly if $\bar{f} = u - iv$ then

$$\boxed{\frac{\partial \bar{f}}{\partial z} = 0} \Rightarrow \bar{f} = \bar{f}(\bar{z})$$

$$\bar{f}(z) = u - iv = \bar{f}(\bar{z}).$$

C_4: $\boxed{\text{Result}}$:- If $f(z) = u + iv$ is analytic s.t. $f(z)$ is purely imaginary or real then it is constant.

i.e. $f \in \mathbb{R}$ or $f \in i\mathbb{R}$

Problem :- Find all the harmonic functions of the form

$u(x,y) = ax^3 + bx^2y + cxy^2 + dy^3$

with minimum unknown co-efficients

OR. Find the most general form of the cubic which is harmonic. Hence or otherwise find $f(z)$ whose real part is u.

Solution :- Consider $u(x,y) = ax^3 + bx^2y + cxy^2 + dy^3$ —— (1)

$$u_x = 3ax^2 + 2bxy + cy^2$$

$$u_{xx} = 6ax + 2by$$

Similarly $u_{yy} = 6dy + 2cx$

For 'u' to be harmonic, we must have

$$u_{xx} + u_{yy} = 0$$

$$\Rightarrow (6a+2c)x + (2b+6d)y = 0$$

$$\Rightarrow 6a = -2c$$

$$\Rightarrow \boxed{c = -3a} \; ; \; \boxed{b = -3d} \quad \text{Put in (1)}$$

$$\boxed{u(x,y) = ax^3 - 3dx^2y - 3axy^2 + dy^3}$$

Which is the most general form of cubic for which it is harmonic.

$$f'(z) = 3ax^2 - 6dxy - 3ay^2 - i(-3dx^2 - 6axy + 3dy^2)$$

$$= 3a(x^2-y^2) + i\,3d(x^2-y^2) - 6xyd + i\,6xya$$

$$f'(z) = 3a(x+iy)^2 + i\,3d(x-iy)^2$$

$$= 3az^2 + i\,3dz^2$$

$$= 3z^2(a+id)$$

$$f(z) = z^3(a+id) + c = (x+iy)^3(a+id) + c$$

$$f(z) = \underbrace{ax^3 - 3axy^2 + dy^3 - 3dx^2y}_{u(x,y)} + i\underbrace{(dx^3 - ay^3 - 3ax^2y - 3dxy^2 + c)}_{\text{Harmonic conjugate of } u = v(x,y)}$$

GATE

Problem:- Find all the harmonic functions of the form $u = \phi(\sqrt{x^2+y^2})$ which are not constant. Hence or otherwise find $f(z)$ whose real part is u.

Solution:- We have

$$u = \phi(\sqrt{x^2+y^2}) \quad \text{---} (1)$$

Here $t = \sqrt{x^2+y^2}$

$$u_x = \phi'(t) \cdot x \cdot \frac{1}{t}$$

$$u_{xx} = \phi''(t) \cdot x^2 \cdot \frac{1}{t^2} + \phi'(t) \cdot \frac{1}{t} - \frac{x^2 \phi'(t)}{t^3}$$

$$u_{yy} = \phi''(t) \cdot y^2 \cdot \frac{1}{t^2} + \phi'(t) \cdot \frac{1}{t} - \frac{y^2 \phi'(t)}{t^3}$$

For (1) to be harmonic, we must have

$$u_{xx} + u_{yy} = 0$$

$$\phi''(t) + 2\frac{\phi'(t)}{t} - \frac{\phi'(t)}{t} = 0$$

$$\phi''(t) + \frac{\phi'(t)}{t} = 0$$

Put $\phi'(t) = v$

$$\Rightarrow \frac{dv}{dt} + \frac{1}{t} v = 0$$

$$\Rightarrow \int \frac{dv}{v} = -\int \frac{1}{t} dt + \ln c$$

$$\Rightarrow \ln v = -\ln t + \ln c$$

$$\Rightarrow \ln v = \ln \frac{c}{t}$$

$$\Rightarrow v = \frac{c}{t}$$

$$\Rightarrow \phi'(t) = \frac{c}{t} \Rightarrow \phi(t) = c \ln t + d$$

$$\Rightarrow \boxed{\phi(\sqrt{x^2+y^2}) = c \log(\sqrt{x^2+y^2}) + d}$$

Which are the required harmonic functions which are not constant.

→ If $c=1$, $d=0$

$u = \frac{1}{2}\log(x^2+y^2)$ is harmc

on $\mathbb{R}-\{0\}$.

→ $f(z) = u + iv$

$= \frac{1}{2}\log(x^2+y^2) + i\tan^{-1}(\frac{y}{x})$

$= \log z + c$

$f(z)$ is analytic in $\mathbb{C}-(-$

Note:- If u is harmonic i

$D \subset \mathbb{R}^2$ and we have obtai

v so that $u+iv$ is analyt

Remember the domain of analyt

of $f(z)$ need not to be D.

It might be proper subset o

D. by above problem

$\boxed{\text{Behaviour of functions at } z=\infty}$:- A function $f(z)$ is said to be analytic at $z=\infty$ if the function $f\left(\frac{1}{z}\right)$ is analytic at the origin.

Similarly, we say that $f(z)$ has a zero of order m, pole of order n and isolated singularity at $z=\infty$ if $f\left(\frac{1}{z}\right)$ has a zero of order m, pole of order n and isolated singularity at the origin respectively. i.e. $z=0$.

Problem :- Which is non-isolated singularity :-

(i) $f(z) = \tan \frac{1}{z-2}$

(ii) $f(z) = \tan\left(1-\frac{2}{z}\right) = \tan\left(\frac{z-2}{z}\right)$

(iii) $f(z) = \sin \frac{1}{z-2}$

(iv) $e^{-(z-2)}$

Solution :- (i) $f(z) = \tan \frac{1}{z-2} = \dfrac{\sin \frac{1}{z-2}}{\cos \frac{1}{z-2}}$

Singularity of $f(z)$ are given by putting

$$\cos \frac{1}{z-2} = 0 = \cos (2n+1)\frac{\pi}{2}$$

$$\Rightarrow z-2 = \frac{2}{(2n+1)\pi}$$

$$\Rightarrow z = 2 + \frac{2}{(2n+1)\pi} \quad ; n \in \mathbb{Z}$$

Let $S = \left\{2 + \frac{2}{(2n+1)\pi} \; ; n \in \mathbb{Z}\right\} \rightarrow$ Isolated singularities.

Limit point of $S = 2$ which is non-isolated essential singularity of $f(z)$.

The behaviour of $f(z)$ at $z=\infty$ is same as that of $f\left(\frac{1}{w}\right)$ at $w=0$.

Now, $f\left(\frac{1}{w}\right) = \tan\left(\frac{1}{\frac{1}{w}-2}\right) = \tan\left(\frac{w}{1-2w}\right)$

At $w=0$, $f\left(\frac{1}{w}\right) = \tan 0 = 0$

So, $f\left(\frac{1}{w}\right)$ has no singularity at $w=0$.

$\Rightarrow$ $f(z)$ " " " " $z=\infty$.

(ii) $f(z) = \tan\left(1-\frac{2}{z}\right)$

(Poles) singularities are obtained by putting

$$\cos\left(1-\frac{2}{z}\right) = 0 = \cos(2n+1)\frac{\pi}{2}$$

$$\Rightarrow \quad 1-\frac{2}{z} = (2n+1)\frac{\pi}{2}$$

$$\Rightarrow \quad \frac{2}{z} = 1-(2n+1)\frac{\pi}{2}$$

$$\Rightarrow \quad z = \frac{4}{2-(n+1)\pi} \rightarrow \text{Isolated singularities.}$$

Limit point of singularities (Poles) $= 0$.

$\Rightarrow$ $z=0$ is essential non-isolated singularity.

$\boxed{\text{At } z=\infty}$: We find behaviour of $f\left(\frac{1}{z}\right)$ at $z=0$

$$f\left(\frac{1}{z}\right) = \tan(1-2z)$$

$\Rightarrow$ 0 is not singularity of $f\left(\frac{1}{z}\right)$

$\Rightarrow$ $z=\infty$ is not singularity of $f(z)$.

(iii) $f(z) = \sin \frac{1}{z-2}$

zeros of $f(z)$ are $\frac{1}{z-2} = n\pi \Rightarrow z-2 = \frac{1}{n\pi}$

$\Rightarrow z = 2 + \frac{1}{n\pi}$

2 is limit point of zeros

$\Rightarrow z=2$ is essential singularity of $f(z)$.

$\boxed{\text{Remember}}$:- $\sin z$ has only singularity at $z=\infty$

So $\sin z$ has isolated ~~singu~~ essential singularity at $z=\infty$

(iv) $f(z) = e^{-(z-2)}$ $\boxed{\text{At } z=\infty}$ put $z = 1/z$ $\frac{1}{e^{(1/z - 2)}} = \frac{1}{e^{1/z} . e^{-2}}$

No singularity $= \frac{e^2}{e^{1/z}} \to z=0$ E.S

But at ∞, it has essential singularity.

$z=\infty$ is essential singularity.

$\boxed{\text{Remarks}}$:-

(i) zeros are isolated.

(ii) Poles are isolated.

(iii) Limit point of zeros is an isolated essential singularity of $f(z)$.

$\boxed{\text{Remember}}$:- If $f(z)$ is analytic then $e^{f(z)}$ is analytic.

$e^{-(z-2)}$ has no singularity but at ∞ it has essential singularity.

(iv) The limit point of a sequence of poles of a function $f(z)$ is non-isolated essential singularity.

Note:- If z_0 is a zero of $f(z)$

$$\Rightarrow \quad f(z_0)=0$$

$$D: \; 0<|z-z_0|<\delta$$

$$f(z)\neq 0 \text{ for any } z\in D$$

$\Rightarrow$ zeros are isolated.

$\rightarrow \quad f(z)= e^{-1/z^2} \; ; \; z\neq 0 \qquad$ (NO singularity)

Problem:- Find zeros of $f(z)= e^{z}-2i$

Solution:- $e^{z} = 2i = e^{\log 2i} = e^{\log 2+(\alpha+2n\pi)i}$

$$e^{z} = e^{\log 2}\cdot e^{(\alpha+2n\pi)i}$$

$$= \log 2+(\alpha+2n\pi)i \quad ; \; n\in\mathbb{Z}$$

		Position	Character
(i) $\sin\frac{1}{z}$ & $\cos\frac{1}{z}$	$z=0$	isolated	essential.
(ii) $\cot\frac{1}{z}$	$z=0$	non-isolated	essential.

→ Poles and removable singularities are isolated singularities by definitions.

→ The limit point of poles infact l.p. of singularities are non-isolated singularities.

$\boxed{\text{e.g}}$:- $\tan\frac{1}{z} = \dfrac{\sin\frac{1}{z}}{\cos\frac{1}{z}}$

$$\cos\frac{1}{z} = 0 \Rightarrow z = \frac{2}{(2n+1)\pi} \to 0$$

$$\langle z_n \rangle \longrightarrow 0$$

$$\sin\frac{1}{z} \longrightarrow 0$$

$$z = \frac{1}{n\pi}$$

Here zero is the l.p. of zeros.

$\boxed{\text{Problems}}$:

(i) $f(z) = \dfrac{1}{\sin z - \cos z}$ → If $f(z) = \dfrac{p(z)}{q(z)}$ → If $q(z)$ are zeros of $p(z)$ then $q(z)$ are poles of $f(z)$

Here $p(z) = 1$

$q(z) = \sin z - \cos z$

$\sin z - \cos z = 0$

$\Rightarrow \tan z = 1$

$\Rightarrow z = \left(n\pi + \frac{\pi}{4}\right)$, $n \in \mathbb{Z}$

→ zeros of $f(z)$

$\Rightarrow$ poles of $f(z)$

& poles ~~are~~ isolated

(ii) $f(z) = \operatorname{cosec}\frac{1}{z}$

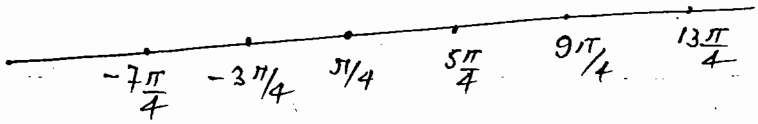
(iii) $f(z) = e^{\tan z}$

(iv) $f(z) = e^{\sin\frac{1}{z}}$

(v) $f(z) = e^{-z}$

Solutions :- (i) We have $\sin z - \cos z = 0$

$\Rightarrow \tan z = 1 \Rightarrow z = n\pi + \frac{\pi}{4}$; $n \in \mathbb{Z}$



Obviously all the singular points $z = n\pi + \frac{\pi}{4}$ are isolated.

Again

$$\lim_{z \to (n\pi + \frac{\pi}{4})} \left[z - \left(n\pi + \frac{\pi}{4}\right)\right] \frac{1}{\sin z - \cos z}$$

$$= \lim_{z \to (n\pi + \frac{\pi}{4})} \left[z - \left(n\pi + \frac{\pi}{4}\right)\right] \frac{1}{\sin z \cos\left(\frac{\pi}{4} + n\pi\right) - \cos z \cdot \sin(n\pi\ldots}$$

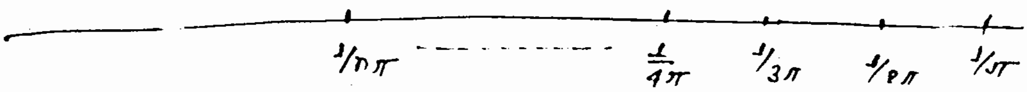
$$= \lim_{z \to (n\pi + \frac{\pi}{4})} \left[z - \left(n\pi + \frac{\pi}{4}\right)\right] \frac{1}{\sin\left[z - \left(n\pi + \frac{\pi}{4}\right)\right]} = 1 \text{ (fi}$$

Hence $z = n\pi + \frac{\pi}{4}$ are simple poles.

(ii) $f(z) = \operatorname{cosec} \frac{1}{z} = \frac{1}{\sin \frac{1}{z}} \longrightarrow \left[\text{Or, } f(z) = \frac{1}{\sin(\pi/z)}\right]$

Let $\sin \frac{1}{z} = 0 \Rightarrow \frac{1}{z} = n\pi$; $n \in \mathbb{Z}$

$\Rightarrow z = \frac{1}{n\pi}$



S = set of all singular points $= \left\{\frac{1}{n\pi} : n \in \mathbb{Z}\right\}$

Limit point of $S = 0$

Hence '0' is the non isolated essential singularity and all other singularities are isolated.

Now, $\lim_{z \to \frac{1}{n\pi}} \left(z - \frac{1}{n\pi}\right) \cdot \frac{1}{\sin \frac{1}{z}}$

$= \lim_{z \to \frac{1}{n\pi}} \frac{1}{\cos 1/z} \cdot \frac{1}{z^2}$

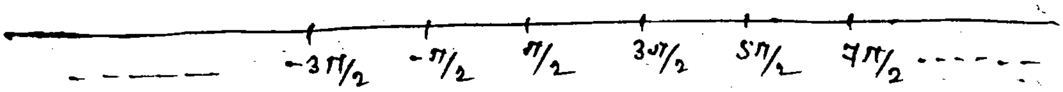
$= \frac{1}{n^2\pi^2} (-1)^{n-1} = \text{finite}$

$\Rightarrow$ All the isolated singularities are simple poles.

Imp (iii) $f(z) = e^{\tan z}$ $\quad (e^{f(z)} \text{ type})$

$= e^{\frac{\sin z}{\cos z}}$

Let $\cos z = 0 \Rightarrow z = (2n+1)\frac{\pi}{2}$; $n \in \mathbb{Z}$



Obviously all the singularities of $f(z)$ are isolated.

Now, $\lim_{z \to (2n+1)\frac{\pi}{2}} e^{\tan z} \left[z - (2n+1)\frac{\pi}{2}\right]$

$= \lim_{z \to (2n+1)\frac{\pi}{2}} \frac{\left[z - (2n+1)\frac{\pi}{2}\right]}{e^{-\tan z}}$ $\quad \to$ Essential

$= \lim_{z \to (2n+1)\frac{\pi}{2}} \frac{1}{e^{-\tan z}(-\sec^2 z)}$

$= - \lim_{z \to (2n+1)\frac{\pi}{2}} \frac{\cos^2 z}{e^{-\tan z}}$

At $z = \infty$ put $z = 1/z$ then $f(z) = e^{\tan 1/z}$ and at z=0

$\therefore z = 0$ is non-isolated E.S

$\Rightarrow$ $\boxed{z = \infty \text{ is non-isolated E.S}}$

(iv) $f(z) = e^{\sin\frac{1}{z}}$ [of the type $e^{f(z)}$] $= e^{\sin f(z)}$

Obviously, $z=0$ is the only singular point (isolated)
Since it is neither removable nor pole.
Therefore $z=0$ is the <u>isolated essential singularity</u> of the function $f(z)$.

(v) $f(z) = e^{-z} = \dfrac{1}{e^z}$

Since $e^z \neq 0$ and entire function, therefore it has no singular point.

Classify the singularities :-

1). $f(z) = (z-3)^{3/2}$. Obviously $z=3$ is a B.P.

2). $f(z) = \log(z^2+z-2) = \log[(z+2)(z-1)]$

Obviously $z=-2, 1$ are B.P.

3). $f(z) = \dfrac{\sin z}{z}$; $z \neq 0$. ($z=0$ is a removable singularity of f.

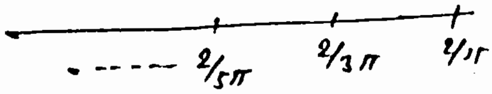
4). $f(z) = e^{\frac{1}{z-2}}$ ($z=2$ is isolated essential singularity)

5). $f(z) = z^3$ has pole of order 3 at ∞.

6). $f(z) = \sec\frac{1}{z} = \dfrac{1}{\cos\frac{1}{z}}$

Let $\cos\frac{1}{z} = 0 \Rightarrow \frac{1}{z} = (2n+1)\frac{\pi}{2}$; $n \in \mathbb{Z}$

$\Rightarrow z = \dfrac{2}{(2n+1)\pi}$; $n \in \mathbb{Z}$



Obviously <u>$z=0$ is non-isolated essential singularity</u> and all other singularities are <u>isolated simple poles.</u>

7). $f(z) = \dfrac{\log(z-2)}{(z^2+2z+2)^2} = \dfrac{\log(z-2)}{(z+1-i)^2\,(z+1+i)^2}$

$\because\ z^2+2z+2=0 \Rightarrow z = \dfrac{-2 \pm \sqrt{4-4\times 2\times 1}}{2} = -1 \pm i$

$\therefore\ z^2+2z+2 = (z+1-i)\,(z+1+i)$

The point $z=2$ is a branch point and is an isolated singularity & $z=-1\pm i$ are poles of order 2 which are isolated singularities.

8). $f(z) = \dfrac{\sin\sqrt{z}}{\sqrt{z}}$ ($\because z=0$ is not a B.P)

$\lim\limits_{z\to 0} \dfrac{\sin\sqrt{z}}{\sqrt{z}} = 1$ (finite)

$\Rightarrow$ $z=0$ is a removable singularity.

9). $f(z) = \sin^{-1}\left(\frac{1}{z}\right)$

Obviously $z=\pm 1$ is B.P. & $z=0$ is an isolated essential singularity.

$f'(z) = \dfrac{1}{\sqrt{1-\left(\frac{1}{z}\right)^2}} \times \left(-\dfrac{1}{z^2}\right) = \dfrac{1}{z\sqrt{z^2-1}} = \dfrac{\frac{1}{\sqrt{z^2-1}}}{z}$

10). $f(z) = \sqrt{z(z^2+1)}$ [$z=0, \pm i$ are the B.P]

11). $f(z) = \dfrac{\cos z}{(z+i)^3}$ [$z=-i$ is an isolated pole of order 3]

Imp
12). $f(z) - e^{z^2}$ [$z=\infty$ is a isolated essential singularity]

13). $f(z) = \operatorname{cosec}\left(\frac{1}{z^2}\right)$

[$z = \dfrac{1}{\sqrt{m\pi}}$; $m = \pm 1, \pm 2, \text{-----}$ simple poles

$z = 0$ is an essential singularity, $z=\infty$ pole of order 2.]

14). $f(z) = \frac{z^2+1}{z^{3/2}}$ [$z=0, \infty$ are B.P].

V.imp
15). $f(z) = e^{1/z^2}$ [$z=0$ is an isolated essential singular
[of type $e^{f(z)}$]

$\boxed{\text{Put } z=0}$

16). $f(z) = \frac{\sinh z}{z^4}$ [$z=0$ is an isolated pole of order 3].

17). $f(z) = \frac{1-\cos z}{z^2}$ [$z=0$ is a removable singularity;

18). $f(z) = e^{1/z}$ [$z=0$ is an isolated essential singularity

19). $f(z) = z\cos\left(\frac{1}{z}\right)$ [$z=0$ is an isolated essential singular.

20). $f(z) = \frac{z-\sin z}{z}$ [$z=0$ is a removable singularity]

21). $f(z) = \frac{\cot z}{z^4}$ [$z=0$ is an isolated pole of order.

22). $f(z) = \frac{e^{-z}}{z^2}$ [$z=0$ is an isolated pole of order 2;

23). $f(z) = z^2 e^{1/z}$ or $z \cdot e^{1/z}$ [$z=0$ is an isolated essential
singularity].

24). $f(z) = \frac{1}{z(e^z-1)}$ [$z=0$ is an isolated pole of order.

25). $f(z) = \frac{\cos z}{\sin z} = \cot z$

[$z=0, \pm\pi, \pm 2\pi, \cdots$ are isolated simple poles
& $z=\infty$ is a non-isolated essential singularity]

26). $f(z) = \tan z$

[$z = (2n+1)\frac{\pi}{2}$; $n = 0, \pm 1, \pm 2, ---$ are isolated simple poles.]

27). $f(z) = \frac{1}{\sinh 2z}$ [$z=0$ is an isolated simple pole].

28). $f(z) = \frac{z^3 \cdot e^{1/z}}{1+z^3}$

[$z=0$ is an isolated essential singularity

& $z = e^{i\pi(2n+1)/3}$; $n = 0, 1, 2 ---$ are isolated simple poles]

29). $f(z) = \frac{1}{z^2 \sin z}$ [$z=0$ is an isolated pole of order 3]

$$\rightarrow \boxed{\operatorname{cosec} z = \frac{1}{z} + \frac{1}{3!} z + \left[\frac{1}{3!} - \frac{1}{5!}\right] z^3 + ---- \quad (0 < |z| < \pi)}$$

30). $f(z) = \frac{1}{z^2} + z^m$

[$z=0$ is an isolated pole of order 2

& $z=\infty$ is an isolated pole of order m]

→ Put $z = \frac{1}{\xi}$ & then see behaviour of $f(z)$ at $\xi = 0$

$\Rightarrow z = \infty$ is a pole of order m].

Imp

31). $f(z) = e^{z}, e^{iz}, e^{-z}$; $\sinh z$, $\cosh z$, $\sin z$, $\cos z$

has a non-isolated essential singularity at $z=\infty$. (38)

32). $f(z) = z^z = e^{z \log z}$

Obviously $z=0$ is the B.P. at $y=0, x<0$ limit of $f(z)$ does not exist and therefore all these singular points become essential singularities.

33). $f(z) = \dfrac{z^3}{\left(2 - \cos\frac{1}{z}\right)}$

$2 - \cos\frac{1}{z} = 0 \Rightarrow \cos\frac{1}{z} = 2 \Rightarrow \frac{1}{z} = \cos^{-1}(2)$

$\therefore \quad z = \dfrac{1}{\cos^{-1}(2)}$

$\Rightarrow z = \dfrac{1}{(\cos^{-1}2)}$ is the isolated simple pole.

Interview
↓
V.Imp → $S = \left\{ 1 + \frac{1}{n} + i \sin\frac{1}{m} \;;\; m, n \in \mathbb{Z} \right\}$

Infinite isolated and infinite non-isolated singularities.

V.Imp → Non-isolated ⇒ Essential singularity

e.g. $f(z) = \tan 1/z$ [

But isolated singularity may be essential

e.g. $f(z) = \sin 1/z$ [$z=0$ is isolated essential singu

Limit points of poles

V.Imp → Limit points of singular points are non-isolated essential singularity.

Result:- If $f(z)$ is analytic in $\mathcal{D}/\{i\}$, where 'i' is limit point of zeros of $f(z)$. Then 'i' is either isolated essential singularity of $f(z)$ or $f(z) \equiv 0$ on $\mathcal{D}$.

Problem:- $f(z) = z^3$. Nature at ∞.

Solution:- $g(z) = \frac{1}{z^3}$

$$z^3 = 0 \Rightarrow z = 0, 0, 0$$

i.e. $z = 0$ is pole of order 3 of $g(z)$

$\Rightarrow$ $z = \infty$ is a pole of order 3 of $f(z)$.

* $\rightarrow$ ~~Entire means there is no singularity at $z=\infty$.~~

Problem:- If $f(z)$ and $g(z)$ are entire &

$$f(1/n) = g(1/n) \quad \forall\, n \in \mathbb{N}$$

then $f(z) = g(z) \;\; \forall\, z \in \mathbb{C}$

Solution:- $h(z) = f(z) - g(z)$

Since $f(z)$, $g(z)$ is analytic

$\Rightarrow$ $h(z)$ is also analytic.

$$h(1/n) = 0 \quad \forall\, n \in \mathbb{N}$$

$S = \{ 1/n$ is set of zeros of h : $n \in \mathbb{N} \}$

$z = 0$ is l.p. of zeros of $h(z)$

But $h(z)$ is analytic at $z = 0$

$\Rightarrow$ $h \equiv 0$

$\Rightarrow$ $h(z) = 0 \;\; \forall\, z \in \mathbb{C}$

$\Rightarrow$ $f(z) - g(z) = 0 \;\; \forall\, z \in \mathbb{C}$ $\Rightarrow$ $f(z) = g(z) \;\; \forall\, z \in \mathbb{C}$.

e.g:- $f(z)$ is entire and $f(z) = \sin^2 z + \cos^2 z - 1$

In real, $\sin^2 z + \cos^2 z = 1$

i.e. $f(z) = 0 \ \forall \ z \in (-\infty, \infty)$

$\mathbb{R}$ is the set of zeros of $f(z)$.

$\therefore$ Every real number is limit point & $f(z)$ is analytic at limit point.

Hence $f(z) \equiv 0 \ \forall z$.

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Q: Result:- If $f(z)$ is analytic and $|f(z) \cdot f'(z)| \le 1$ then $f(z)$ is constant.

Solution:- Given $|f(z) \cdot f'(z)| \le 1$

Let $g(z) = \frac{1}{2}[f(z)]^2 \Rightarrow f(z) = \sqrt{2g(z)}$

$g'(z) = f(z) f'(z)$

$\Rightarrow |g'(z)| = |f(z) f'(z)| \le 1$

$\Rightarrow |g'(z)| \le 1$

$\Rightarrow g'(z)$ is entire and bounded.

So by Liouville's theorem,

$g'(z)$ is constant.

$\Rightarrow g'(z) = a$ (say)

$\Rightarrow g(z) = az + b$

$\therefore f(z) = \sqrt{2(az+b)}$

$f(z)$ has B.P. at $az+b=0 \Rightarrow \boxed{z = -b/a}$

$f(z)$ is analytic. So $f(z)$ fails to have B.P. if $a=0$

$\Rightarrow f(z) = \sqrt{2b} \Rightarrow \boxed{f(z) = c} \rightarrow$ Hence proof.

Result:- If $f(z) = u+iv$ is analytic in $\mathbb{C}$ and u, v lie on a parabola (any type) then $f(z)$ is constant.

Proof:- Let $v^2 = u$, $u^2 = v$

diff. partially, we have

$$2vv_x = u_x \quad \text{---(I)}$$

$$2vv_y = u_y \quad \text{---(II)}$$

By C-R equation

From (I)

$$-2vu_y = u_x$$

From (II)

$$2vu_x = u_y$$

$$\Rightarrow 2v(-2vu_y) = u_y$$

$$\Rightarrow -4v^2 u_y = u_y$$

$$\Rightarrow (4v^2+1)u_y = 0$$

$$\Rightarrow \boxed{u_y = 0}$$

Similarly $\boxed{u_x = 0}$

Hence $f(z) = u+iv$ is constant.

Note:- $z = x+iy$, $\bar{z} = x-iy$

$$\boxed{x = \frac{z+\bar{z}}{2}} \quad , \quad \boxed{y = \frac{z-\bar{z}}{2i}}$$

$$\boxed{\frac{\partial x}{\partial z} = \frac{1}{2}} \quad , \quad \boxed{\frac{\partial y}{\partial z} = \frac{1}{2i}}$$

$$\boxed{\frac{\partial x}{\partial \bar{z}} = \frac{1}{2}} \quad , \quad \boxed{\frac{\partial y}{\partial \bar{z}} = -\frac{1}{2i}}$$

NOW, $u = u(x,y)$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial z} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial z}$$

$$= \frac{1}{2}u_x + \frac{1}{2i}u_y$$

$$\frac{\partial^2 u}{\partial \bar{z}\,\partial z} = \frac{1}{2}\frac{\partial}{\partial \bar{z}}(u_x) + \frac{1}{2i}\frac{\partial}{\partial \bar{z}}(u_y)$$

$$= \frac{1}{2}\left[\frac{\partial^2 u}{\partial x^2}\cdot\frac{1}{2} - \frac{\partial^2 u}{\partial x\partial y}\cdot\frac{1}{2i}\right] + \frac{1}{2i}\left[\frac{\partial^2 u}{\partial x\partial y}\cdot\frac{1}{2} - \frac{\partial^2 u}{\partial y^2}\cdot\frac{1}{2}\right]$$

$$= \frac{1}{4}\left[(u_{xx} + i u_{xy}) - i(u_{xy} + i u_{yy})\right]$$

$$\boxed{\frac{\partial^2 u}{\partial \bar{z}\,\partial z} = \frac{1}{4}\left[u_{xx} + u_{yy}\right]}$$

$$\boxed{\frac{\partial^2 u}{\partial \bar{z}\,\partial z} = \frac{1}{4}\nabla^2 u}$$

i.e. $$\boxed{\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4\frac{\partial^2}{\partial \bar{z}\,\partial z}}$$ → complex form of Laplacian operator.

Problem:. Let $f(z)$ be analytic then show that

$$\nabla^2 |f(z)|^2 = 4|f'(z)|^2$$

Solution:- L.H.S $= \nabla^2 |f(z)|^2$

$$= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|f(z)|^2$$

$$= 4\frac{\partial^2}{\partial \bar{z}\,\partial z}\left[|f(z)|^2\right]$$

$$= 4\frac{\partial^2}{\partial \bar{z}\,\partial z}\left[f(z)\,f(\bar{z})\right]$$

$$= \frac{4\,\partial}{\partial \bar{z}}\left[\frac{\partial}{\partial z}\left(f(z)\,f(\bar{z})\right)\right]$$

$$= 4\frac{\partial}{\partial \bar{z}}\left[f'(z)\cdot \bar{f}(\bar{z})\right]$$

$$= 4\, f'(z)\cdot \bar{f}'(\bar{z})$$

$$= 4 f'(z)\cdot \overline{f'(z)}$$

$$= 4\,|f'(z)|^2 = R.H.S.$$

Hence $\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right]|f(z)|^2 = 4\,|f'(z)|^2$ Proved

$\rightarrow$ $\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|\text{Re}\, f(z)|^2 = 2\,|f'(z)|^2$

$\rightarrow$ $f(z) = \sin x \cosh y - i\cos x \sinh y = \sin\bar{z}$

[not analytic & have no any singularity]

V.Imp

$\rightarrow$ (i) $A = \left\{\lim_{z\to z_0} f(z_n) : \langle z_n\rangle \to z_0\right\} \to$ Removable singularity then

$\Rightarrow |A| = \{l\}$

(ii) $B = \left\{\lim_{z\to z_0} f(z_n) : \langle z_n\rangle \to z_0\right\} \to$ Pole then

$|B| = \{\infty\}$

(iii) $C = \left\{\lim_{z\to z_0} f(z_n) : \langle z_n\rangle \to z_0\right\} \to$ Essential singularities then

$|C| = \mathbb{C}$

Construction of analytic function :-

Method I :- **Differential equation method** :-

$u = u(x,y)$ (given)

to find $f(z) = u + iv$ s.t. $f(z)$ is analytic upto a constant

Case I :- If u is not harmonic then no such $f(z)$ exist

Case II :- If u is harmonic then we proceed as follows :-

Let v, becomes $f(z)$ is analytic

$$dv = v_x\,dx + v_y\,dy$$

$$\Rightarrow \boxed{dv = -u_y\,dx + u_x\,dy}$$

$$\boxed{\int dv = -\int u_y\,dx + \int u_x\,dy}$$

Check exact

$$-u_{yy} = u_{xx}$$

$$\boxed{u_{xx} + u_{yy} = 0} \quad \text{i.e} \quad \boxed{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0}$$

Problem :- $u = 4xy - x^3 + 3xy^2$

Find v so that $u + iv$ is analytic.

Hence or otherwise find $f(z)$.

Solution :- Now, $\frac{\partial^2 u}{\partial x^2} = -6x$, $\frac{\partial^2 u}{\partial y^2} = 6x$

Hence $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

i.e. u is harmonic.

Now, $dv = -u_y\,dx + u_x\,dy$

$$\Rightarrow dv = -(4x + 6xy)\,dx + (4y - 3x^2 + 3y^2)\,dy$$

$$\Rightarrow dv = -4x\,dx - 6xy\,dx - 3x^2\,dy + 4y\,dy + 3y^2\,dy$$

$$\int dv = \int -4x\,dx - \int d(3x^2 y) + \int 4y\,dy + \int 3y^2\,dy$$

~~Method III~~ : ~~Milne-Thomson Method~~ :-

Now, $f(z) = u + iv$

$$\boxed{f(z) = 4xy - x^3 + 3xy^2 + i\,(-2x^2 - 3x^2y + 2y^2 + y^3 + c)}$$

We know that

$$\boxed{f(z) = 2u\left(\frac{z}{2}, \frac{z}{2i}\right) - u(0,0) + iC}$$

i.e. $f(z) = 2\left[4\cdot\left(\frac{z}{2}\cdot\frac{z}{2i}\right) - \left(\frac{z}{2}\right)^3 + 3\left(\frac{z^2}{4i^2}\cdot\frac{z}{2}\right)\right] - 0 + iC$

$$= 2\left[-iz^2 - \frac{z^3}{8} - \frac{3}{8}z^3\right] + iC$$

$$= -\frac{2}{8}\left[8iz^2 + z^3 + 3z^3\right] + iC$$

$$\boxed{f(z) = -\frac{1}{4}\left[4z^3 + 8iz^2\right] + iC} \rightarrow \text{Method II.}$$

and Milne-Thomson method.

Method II :- Milne-Thomson Method :-

(Necessary and sufficient condition of analyticity)

If $f(z) = u + iv$ is analytic & u is given, then

as we know

$$f'(z) = u_x + iv_x$$

$$= \underline{u_x - iu_y} \quad \text{v. imp}$$

$$= \phi(z).$$

OR, $f'(z) = u_x(x,y) + iv_x(x,y)$

$\Rightarrow f'(z) = u_x(x,y) - iu_y(x,y)$

put $x = z,\ y = 0$

$f'(z) = u_x(z,0) - iu_y(z,0)$

$z = x + iy$

put $y = 0$

$x = z$

$\Rightarrow f'(z) = \phi_1(z,0) - i\phi_2(z,0)$

$\Rightarrow \boxed{f(z) = \int [\phi_1(z,0) - i\phi_2(z,0)]\,dz + C}$ where $C \in \mathbb{C}$

$\boxed{\text{e.g.}}$:- Consider the function $u = u(x,y) = 2xy + 2x$

$\therefore f'(z) = u_x - iu_y$

$= 2y + 2 - i(2x) = 2y - 2ix + 2$

$= -2i(x + iy) + 2$

$f'(z) = -2iz + 2.$

Integrating, we get

$f(z) = -iz^2 + 2z + c$

Where 'c' is some complex constant.

Now, $f(z) = u + iv$

$\Rightarrow$ harmonic conjugate of $u(x,y) = v(x,y) = y^2 + 2y - x^2$

$\Big[\because f(z) = -i(x+iy)^2 + 2(x+iy) + c$

$= -i(x^2 - y^2 + 2ixy) + 2x + 2iy + c$

$= 2xy + 2x + i(y^2 - x^2 + 2y) + c\Big]$

Method III :- If $f: D \to \mathbb{C}$

If f is analytic $\Rightarrow \frac{\partial f}{\partial \bar{z}} = 0$

$f(z) = u + iv$

$f(x+iy) = u(x,y) + iv(x,y)$

$\bar{f}(x-iy) = u(x,y) - iv(x,y)$

Now, $\bar{f}(0) = u(0,0) - iv(0,0)$

Again $\bar{f}(0) = u(0,0) - i\underbrace{C}_{\text{Real}}$ ——— (1)

Now, $2u(x,y) = f(x+iy) + \bar{f}(x-iy)$

$2u\left(\frac{z}{2}, \frac{z}{2i}\right) = f\left(\frac{z}{2} + \frac{z}{2}\right) + \bar{f}\left(\frac{z}{2} - \frac{z}{2}\right)$

$x = \frac{z+\bar{z}}{2}$, $y = \frac{z-\bar{z}}{2i}$

But Here f is independent of $\bar{z}$ i.e. $\bar{z} = 0$

$\therefore x = \frac{z}{2}$, $y = \frac{z}{2i}$

$\Rightarrow 2u\left(\frac{z}{2}, \frac{z}{2i}\right) = f(z) + \bar{f}(0)$

By equation (1)

$2u\left(\frac{z}{2}, \frac{z}{2i}\right) = f(z) + u(0,0) - iC$

or. $\boxed{f(z) = 2u\left(\frac{z}{2}, \frac{z}{2i}\right) - u(0,0) + iC}$

and $\boxed{f(z) = i2v\left(\frac{z}{2}, \frac{z}{2i}\right) - iv(0,0) + C}$

Remarks :-

→ Here v can be infinite in numbers w.r.t. u.

→ This does not give all the analytic function where real part is u but gives analytic function upto real constant.

→ We should take care of origin as $(0,0)$.

Problem 7:- If $u = ax^3 + bx^2y + cxy^2 + dy^3$, Find $f(z)$?

Solution:- Put $x = \frac{z}{2}$, $y = \frac{z}{2i}$, we get

$$f(z) = 2\left[\frac{az^3}{8} + b\frac{z^2}{4}\cdot\frac{z}{2i} + c\cdot\frac{z}{2}\cdot\frac{z^2}{4i^2} + d\frac{z^3}{8i^3}\right]$$

$$= \frac{2}{8_4}\left[az^3 - ibz^3 - cz^3 + idz^3\right] + iC'$$

$$= \frac{1}{4}z^3\left[a - ib - c + id\right] + iC'$$

$$\Rightarrow \boxed{f(z) = \frac{1}{4}z^3\left[(a-c) + i(d-b)\right] + iC'}$$

or, $f(z) = \frac{1}{4}z^3\left[4a + i4d\right] + iC'$

$$\Rightarrow \boxed{f(z) = z^3(a+id) + iC'}$$

Note:- This method fails when singularity at $z=0$ i.e. at $(0,0)$ then $f(0)$ is not defined.

e.g.: If $u = \frac{1}{2}\log(x^2+y^2)$ then we use other method.

e.g.:- $f(z) = \log z + C$

$$= \frac{1}{2}\log(x^2+y^2) + \left[\tan^{-1}\left(\frac{y}{x}\right)\right]i + C.$$

$u = \frac{1}{2}\log(x^2+y^2)$ is harmonic on $\mathcal{D} - \{0\}$

i.e. $f(z)$ is analytic in $\mathbb{C} - (-\infty, 0]$.

Note:- If u is harmonic on $\mathcal{D}$ then $\exists$ v s.t. $u+iv$ is analytic but domain of analyticity of $f(z)$ need not to be D but it is contained in D & v so obtained is <u>unique</u> <u>upto contant</u>.

Note (i) Removable singularity, poles, branch point are isolated singularity.

(ii) Essential singularity —
- → Isolated singularity
- → Non-isolated singularity.

~~Branch point may be non-isolated, if singularities in nbd of branch point.~~

~~**Branch point** :- $z=z_0$ is a branch point $\exists\ \delta$ s.t. $0<|z-z_0|<\delta$ and there is no other branch point of $f(z)$ but may have singularity of other type. Then if there is singularity in nbd of $z=z_0$ then the branch point $z=z_0$ is non isolated singularity.~~

Pole or Removable Singularity :- $z=z_0$ is a pole or removable singularity $\exists\ \delta$ s.t. $0<|z-z_0|<\delta$ and there is no other pole or singularity of $f(z)$.

→ Pole or Removable singularity is always isolated.

Note :- Limit point of singularities is always non-isolated essential singularity.

i.e. Limit points of poles are non-isolated essential singularity.

Problem:- Discuss the singularities of the function

$$f(z) = \frac{\sin z}{z^5(z-2)} e^{\frac{1}{z-1}}$$

Solution:

$z=0$ is a pole of order 4

$z=1$ is essential singularity

$z=2$ is a pole of order 1.

$z=\infty$ is essential singularity

Isolated singular

At $z=0$:-

$$\lim_{z\to 0} \frac{z^4}{z^4}\left(\frac{\sin z}{z}\right) \frac{1}{z-2} e^{\frac{1}{z-1}}$$

$$= 1 \cdot \frac{1}{-2} \cdot e^{-1} = \frac{-1}{2e} \Rightarrow$$ limit exists, nonzero and finite.

So $z=0$ is a pole of order 4.

At $z=1$:- No pole, No removable. So essential.

At $z=2$:- $\lim_{z\to 2} (z-2) \frac{\sin z}{z^5(z-2)} e^{\frac{1}{z-1}}$

$$= \frac{\sin 2}{2^5} e^1 = \frac{e \sin 2}{2^5} \Rightarrow$$ limit exists, non.. and finite

So $z=2$ is a pole of order 1 i.e. $z=2$ is a simple pc

At $z=\infty$:- To check the behaviour of $f(z)$ at $z=\infty$ we change $z=\frac{1}{z}$ and check at $z=0$.

$$\lim_{z\to 0} f(1/z) = \lim_{z\to 0} \underbrace{\sin 1/z}_{\text{limit does not exist}} \cdot e^{\frac{z}{1-z}} \cdot \frac{z^6}{(1-2z)} \Rightarrow$$ Limit does not exist

$\Rightarrow z=\infty$ is essential singularity.

Problem:- $f(z) = \dfrac{z+3}{z^2-1}$

Solution:- Now, $g(z) = f(1/z) = \dfrac{z(1+3z)}{1-z^2}$

$z=0$ is a regular point of $g(z)$

then $z=\infty$ is a regular point of $f(z)$.

Imp

Problem:- $f(z) = \sqrt{\dfrac{z+3}{z^2-1}}$ (Ask) पूछना है

Solution:- $w = \frac{1}{2}\arg(z+3) - \frac{1}{2}\arg(z^2-1)$

$z=-3$ is a branch point

Now, taking principal branch

$f(z) = \left(\dfrac{z+3}{z^2-1}\right)^{1/2}$ is a single valued function

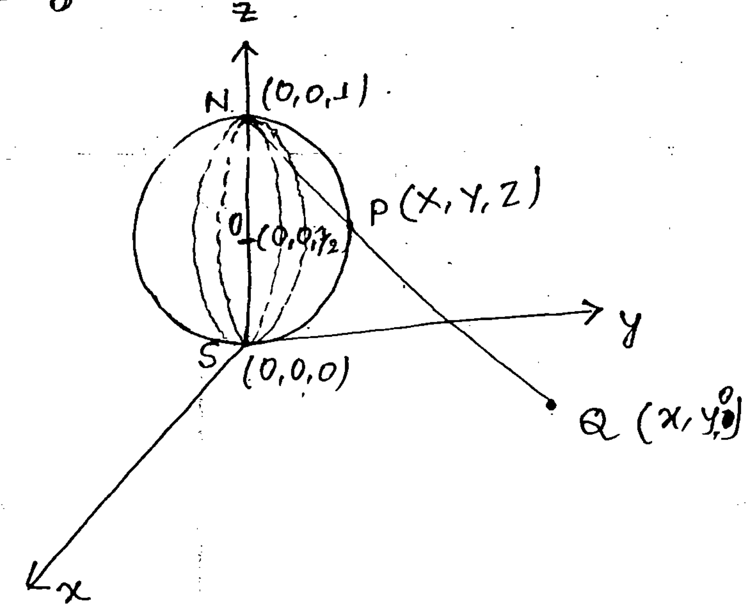
Now, $\lim\limits_{z\to 1} \dfrac{\cancel{(z-1)^{1/2}}\,(z+3)^{1/2}}{\cancel{(z-1)^{1/2}}\,(z+1)^{1/2}} = \dfrac{4^{1/2}}{2^{1/2}} = 2^{1/2}$

$= \sqrt{2} \neq 0$ (finite)

$\boxed{\text{But } \frac{1}{2} \notin N}$

$\therefore$ $\boxed{z=1 \text{ is not pole}}$

Stereographic Projection (V.V. Imp)



Let S be the sphere centre at $(0, 0, 1/2)$ and radius is $1/2$ unit and diameter is 1 and $N(0,0,1)$ the point on z-axis is represent at north pole and origin $S(0,0,0)$ is south pole.

Let Q be a point $x+iy$ in xy-plane if we joint Q to the north pole N the line segment QN intersect the sphere in $P(X,Y,Z)$ then this P is unique for each Q and conversely.

i.e. We take any point on sphere and join to the point of north pole their extension of this line segment meet the xy-plane at a unique point.

Hence we can conclude for each point in xy-plane and conversely and the process of binding such correspondence with the choosen sphere to the complex plane is called Stereographic projection

$NQ : \{ t(0,0,1) + (1-t)(x,y,0) \} \quad : \quad 0 \le t \le 1$

$= \{ (1-t)x, \ (1-t)y, \ t \}$

Equation of sphere $X^2+Y^2+(Z-\frac{1}{2})^2 = \frac{1}{4}$ —— (1)

and the plane of projection is $z=0$ i.e. (xy plane)

We take $N=(0,0,1)$ corresponds to point ∞ in xy-plane. For any point $P=(X,Y,Z)$ on the sphere. We have point $Q=(x,y,0)$ in the complex plane where NQ meets the plane of projection. Since the points $(0,0,1)$, (X,Y,Z), $(x,y,0)$ are on the straight line, we have

$$\frac{X-0}{x-0} = \frac{Y-0}{y-0} = \frac{Z-1}{0-1} = t \Rightarrow \boxed{X = xt, \ Y = yt, \ Z = 1-t}$$

$\because$ Equation of line passing through two points (x_1, y_1, z_1) & (x_2, y_2, z_2)

$$\frac{X-x_1}{x_2-x_1} = \frac{Y-y_1}{y_2-y_1} = \frac{Z-z_1}{z_2-z_1}$$

$$S : (X-0)^2 + (Y-0)^2 + (Z-\tfrac{1}{2})^2 = \tfrac{1}{4}$$

$$\Rightarrow X^2 + Y^2 + (Z-\tfrac{1}{2})^2 = \tfrac{1}{4}$$

$$(1-t)^2x^2 + (1-t)^2y^2 + (t-\tfrac{1}{2})^2 = \tfrac{1}{4}$$

Find $t = ?$

$$x^2t^2 + y^2t^2 + (1-t-1/2)^2 = 1/4$$

$$\Rightarrow x^2t^2 + y^2t^2 + \tfrac{1}{4} + t^2 - t = 1/4$$

$\Rightarrow t^2(x^2+y^2+1) - t = 0$

$\Rightarrow t\,[t\,(x^2+y^2+1) - 1] = 0$

$\Rightarrow \quad t = 0 \quad$ or $\quad t = \dfrac{1}{x^2+y^2+1}$

$\rightarrow$ If $t=0$ then $(X, Y, Z) = (0, 0, 1)$ i.e. North pole.

$\rightarrow$ If $t = \dfrac{1}{x^2+y^2+1}$ then

$$\boxed{\begin{aligned} X &= \frac{x}{x^2+y^2+1} = \frac{z+\bar{z}}{2(1+|z|^2)} \\ Y &= \frac{y}{x^2+y^2+1} = \frac{z-\bar{z}}{2i(1+|z|^2)} \\ Z &= \frac{x^2+y^2}{x^2+y^2+1} = \frac{|z|^2}{1+|z|^2} \end{aligned}}$$

and $\boxed{X = xt, \quad Y = yt, \quad Z = 1-t}$

$$\boxed{x = \frac{X}{t}, \quad y = \frac{Y}{t}, \quad \begin{aligned} Z &= 1-t \\ t &= 1-Z \end{aligned}}$$

then $\boxed{x = \dfrac{X}{1-Z}}$ $\quad \boxed{y = \dfrac{Y}{1-Z}}$ $\quad \boxed{z = \dfrac{X}{1-Z} + i\dfrac{Y}{1-Z}}$

Problem:- Find the projection of following point in xy-plane onto sphere with centre $(0, 0, 1/2)$ and north pole is $(0, 0, 1)$

(i) $z = 1$ (ii) $z = i$ (iii) $z = \frac{1-i}{\sqrt{2}}$ (iv) $z = \frac{1+i}{\sqrt{2}}$

Solution (i). $\boxed{z = 1}$

$$X = \frac{z+\bar{z}}{2(1+|z|^2)} = \frac{1+1}{2(1+1)} = \frac{1}{2}$$

$$Y = \frac{z-\bar{z}}{2i(1+|z|^2)} = \frac{1-1}{2i(1+1)} = 0$$

$$Z = \frac{|z|^2}{1+|z|^2} = \frac{1}{1+1} = \frac{1}{2}$$

i.e. $(X, Y, Z) = (1/2, 0, 1/2)$ Ans

(ii) $\boxed{z = i}$

$X = 0$, $Y = 1/2$ $Z = 1/2$

i.e. $(X, Y, Z) = (0, 1/2, 1/2)$ Ans

(iii) $\boxed{z = \frac{1-i}{\sqrt{2}}}$

Now, $|z| = \sqrt{\frac{1}{2}+\frac{1}{2}} = 1$

$X = \frac{\sqrt{2}}{2(1+1)} = \frac{1}{2\sqrt{2}}$, $Y = \frac{-\sqrt{2}i}{2i(1+1)} = -\frac{1}{2\sqrt{2}}$, $Z = \frac{1}{1+1} = \frac{1}{2}$

i.e. $(X, Y, Z) = (1/2\sqrt{2}, -1/2\sqrt{2}, 1/2)$ Ans

(iv) $z = \frac{1+i}{\sqrt{2}}$; $|z| = 1$

$X = \frac{\sqrt{2}}{2\cdot 2} = \frac{1}{2\sqrt{2}}$, $Y = \frac{1}{2\sqrt{2}}$, $Z = \frac{1}{2}$

i.e. $(X, Y, Z) = \left(1/2\sqrt{2}, \frac{1}{2\sqrt{2}}, \frac{1}{2}\right)$ Ans

Chordal distance :- Let stereographic projection of z_1 and are S_1 and S_2 on the sphere, then the distance between the point S_1 and S_2 is defined as the chordal distance betwe z_1 and z_2 and denoted by

$$d(z_1, z_2) = \psi$$

$$d(z, \infty) = \quad z \longrightarrow (X, Y, Z)$$

$$\infty \longrightarrow (0, 0, 1)$$

$$= \sqrt{(X-0)^2 + (Y-0)^2 + (Z-1)^2}$$

$$= \sqrt{\frac{x^2}{(x^2+y^2+1)^2} + \frac{y^2}{(x^2+y^2+1)^2} + \frac{1}{(x^2+y^2+1)^2}}$$

$$= \frac{1}{\sqrt{x^2+y^2+1}}$$

i.e $\boxed{d(z, \infty) = \frac{1}{\sqrt{1+|z|^2}}}$

$$\boxed{d(z_1, z_2) = \frac{|z_1 - z_2|}{\sqrt{1+|z_1|^2}\sqrt{1+|z_2|^2}}}$$

Chordal distance.

e.g:- (i) $d(i, \infty) = \frac{1}{\sqrt{1+1}} = \frac{1}{\sqrt{2}}$

(ii) $d(1, -1) = \frac{|1+1|}{\sqrt{1+1}\sqrt{1+1}} = 1$

Problem:- Find the stereographic projection of i

Solution:- $z = x+iy = i$

$x=0$
$y=1$

$$X = \frac{0}{0+1+1} = 0, \quad Y = \frac{1}{0+1+1} = \frac{1}{2}, \quad Z = \frac{1}{0+1+1} = \frac{1}{2}$$

i.e. $(X, Y, Z) = (0, 1/2, 1/2)$

Problem:- Find the chordal distance of i & ∞.

Solution:- Stereographic projection of $i = (0, 1/2, 1/2)$

" " " $\infty = (0, 0, 1)$

$$\text{Chordal distance} = \sqrt{(0-0)^2 + (1/2-0)^2 + (1/2-1)^2}$$

$$= \sqrt{0 + \frac{1}{4} + \frac{1}{4}}$$

$$= \frac{1}{\sqrt{2}}$$

Problem:- Find the chordal distance between z & ∞.

Solution:- Stereographic projection of $z = \left(\frac{x}{x^2+y^2+1}, \frac{y}{x^2+y^2+1}, \frac{x^2+y^2}{x^2+y^2+1}\right)$

" " " $\infty = (0, 0, 1)$

$$\text{Chordal distance } d(z, \infty) = \sqrt{\left(\frac{x}{x^2+y^2+1} - 0\right)^2 + \left(\frac{y}{x^2+y^2+1} - 0\right)^2 + \left(\frac{x^2+y^2}{x^2+y^2+1} - 1\right)^2}$$

$$= \sqrt{\frac{x^2}{(x^2+y^2+1)^2} + \frac{y^2}{x^2+y^2+1} + \frac{1}{x^2+y^2+1}}$$

$$= \sqrt{\frac{x^2+y^2+1}{(x^2+y^2+1)^2}} = \frac{1}{\sqrt{x^2+y^2+1}} = \frac{1}{\sqrt{1-Z}}$$

$$\left[\because Z = \frac{x^2+y^2}{x^2+y^2+1}, \quad 1-Z = 1 - \frac{x^2+y^2}{x^2+y^2+1}\right.$$

$$\left. = \frac{1}{x^2+y^2+1}\right]$$

Diametrically opposite points (Antipodal points) :-

If chordal distance between z_1 and z_2 is 1 then we say that the points are diametrically opposite

i.e. the correspondence of z_1 & z_2 are the extremities of diameter of the sphere then z_1 and z_2 are diametrically opposite points.

$$d(z_1, z_2) = \frac{|z_1 - z_2|}{\sqrt{1+|z_1|^2}\sqrt{1+|z_2|^2}}$$

$$d\left(\frac{1}{z_1}, \frac{1}{z_2}\right) = \frac{\left|\frac{1}{z_1} - \frac{1}{z_2}\right|}{\sqrt{1+\frac{1}{|z_1|^2}}\sqrt{1+\frac{1}{|z_2|^2}}}$$

$$= \frac{|z_2 - z_1| \cancel{z_1 z_2}}{\cancel{z_1 z_2}\sqrt{1+|z_1|^2}\sqrt{1+|z_2|^2}}$$

$$= \frac{|z_2 - z_1|}{\sqrt{1+|z_1|^2}\sqrt{1+|z_2|^2}}$$

$$d\left(\frac{1}{z_1}, \frac{1}{z_2}\right) = d(z_1, z_2)$$

→ Points are diametrically opposite if $\boxed{d(z_1, z_2) = 1}$

$$d(z_1, z_2) = \frac{|z_1 - z_2|}{\sqrt{1+|z_1|^2}\sqrt{1+|z_2|^2}} = 1$$

$$\Rightarrow |z_1 - z_2| = \sqrt{1+|z_1|^2}\sqrt{1+|z_2|^2}$$

$$\sqrt{(z_1 - z_2)(\overline{z_1 - z_2})} = \sqrt{1+|z_1|^2}\sqrt{1+|z_2|^2}$$

$$\Rightarrow (z_1 - z_2)(\bar{z}_1 - \bar{z}_2) = (1 + z_1\bar{z}_1)(1 + z_2\bar{z}_2)$$

$$\Rightarrow \boxed{z_1\bar{z}_2 = -1} \quad \text{or} \quad \boxed{\bar{z}_1 \cdot z_2 = -1} \quad ; \underline{z_i \neq 0, \infty}$$

This is the condition on z_1 & z_2 s.t. z_1 and z_2 are diametrically

Note: (i) 0 is the antipodal of ∞ and conversely.

(ii) If z_1 and z_2 are antipodal then $\frac{1}{z_1}$ & $\frac{1}{z_2}$ are also antipodal.

Problem:- Find on condition on α, β, γ and α', β', γ' so that when z corresponds to α, β, γ, $i(z)$ corresponds to α', β', γ'.

Solution:- z corresponds to α, β, γ

$$\Rightarrow \quad \alpha = \frac{x}{x^2+y^2+1}$$

$$\beta = \frac{y}{x^2+y^2+1}$$

$$\gamma = \frac{x^2+y^2}{x^2+y^2+1}$$

$i(z)$ corresponds to α', β', γ' $\quad [iz = ix - y \Rightarrow iz = -y + ix]$

$$\alpha' = \frac{-y}{x^2+y^2+1} = -\beta$$

$$\beta' = \frac{x}{x^2+y^2+1} = \alpha$$

$$\gamma' = \frac{x^2+y^2}{x^2+y^2+1} = \gamma.$$

Note:- Stereographic projection of $i = (0, 1/2, 1/2)$

" " " $-1 = i \cdot i = (-1/2, 0, 1/2)$.

→ Problem :- $|z-i| \leq 1$, then which of the following is answer such that $f(z)$ is analytic.

(a) $u = x^2+y^2$ (not harmonic)

(b) $u = e^{xy}$ (not harmonic)

✓(c) $u = \log(x^2+y^2)$ (harmonic) Real part of $f(z)$.

(d) $u = e^{x^2-y^2}$ (not harmonic)

→ Problem :- Choose u s.t. $f(z)$ is analytic in $|z| \leq r$.

(a) $u = x^2-y^2$

(b) $u = \frac{1}{2}\log(x^2+y^2)$

(c) $u = \frac{x}{x^2+y^2}$

(d) $u = e^{\frac{x}{x^2+y^2}} \cdot \cos\frac{y}{x^2+y^2}$

Ans :- In $|z-i|$, all u are harmonic.

S.K. RATHORE

UNIT-2

COMPLEX INTEGRATION

Curve or Path:- A continuous curve or path in $\mathbb{C}$ is a continuous map γ from $[a,b] \subset \mathbb{R}$ into $\mathbb{C}$

i.e. $\gamma : [a,b] \longrightarrow \mathbb{C}$

$$\gamma(t) = x(t) + iy(t) \quad ; t \in [a,b]$$

as γ is continuous path or curve, $x(t)$ and $y(t)$ are continuous functions of t and $\gamma(a)$ & $\gamma(b)$ are called terminal points.

e.g.:- Let $x = \frac{1}{t}$, $y = t$; $t \in [1,3]$

$$xy = 1$$

$$\gamma(t) = x(t) + iy(t)$$

and if $P = \{a = t_0, t_1, t_2, \ldots t_n = b\}$

then for each t_i, $\gamma(t_i)$ denote a complex number

viz. $\gamma(t_i) = z_i = z(t_i)$

$$\boxed{\gamma'(t_i)dt = dz}$$

Jordan Arc:- If $\gamma(t)$ is a continuous curve. $t \in [a,b]$ and for any $t_1 < t_2$, $\gamma(t_1) \neq \gamma(t_2)$ $\forall$ $t_1, t_2 \in \underline{(a,b)}$ then the path is called Jordan arc.

i.e. $\gamma(t)$ is 1-1 on $t \in (a,b)$ then $\gamma(t)$ is defined as Jordan arc

e.g.:- $\gamma(t) = e^{it} = \cos t + i\sin t$, $t \in [0,\pi]$

is Jordan arc.

Jordan Curve:- If $\gamma(t)$ is a Jordan arc on $[a,b]$ s.t. $\gamma(a)=\gamma(b)$ then it is called Jordan curve.

e.g.:- $\gamma(t) = e^{it}$
$= \cos t + i\sin t$

$t \in [0,\pi] \longrightarrow$ Jordan arc
$t \in [0,2\pi] \longrightarrow$ Jordan curve
$t \in [0,3\pi] \longrightarrow$ Neither Jordan arc nor Jordan curve.

$x = \cos t, \quad y = \sin t$

$x^2+y^2 = 1$

Closed Curve:- If $\gamma : [a,b] \longrightarrow \mathbb{C}$

as $\gamma(t) = x(t) + iy(t)$

If $\gamma(a) = \gamma(b)$ then γ is closed.

i.e. when the terminal point coincide, we say curve is closed curve.

i.e $\gamma(a) = \gamma(b)$.

Note (i) Every Jordan curve is closed curve but converse is not true.

i.e. Jordan curve $\Rightarrow$ closed curve.

(ii) Closed curve is called Jordan curve if it is Jordan arc.

e.g.:- If $\gamma(t) = e^{it}$ on $[0, 4\pi]$

Since $\gamma(0) = \gamma(4\pi)$

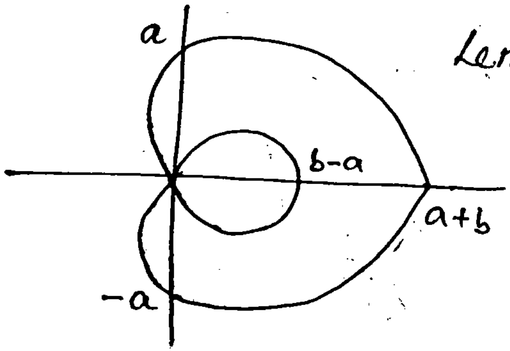
So it is closed but not Jordan curve.

($\because$ not 1-1 so no Jordan curve)

$\gamma(t)$	t	Jordan arc	Jordan curve	Closed
e^{it}	$[0,\pi]$	✓	✗	✗
e^{it}	$[0,2\pi]$	✓	✓	✓
e^{it}	$[0,3\pi]$	✗	✗	✗
e^{it}	$[0,4\pi]$	✗	✗	✓

Note:- $r = a + b\cos\theta$; $\theta \in [0, 2\pi]$.
$a < b$

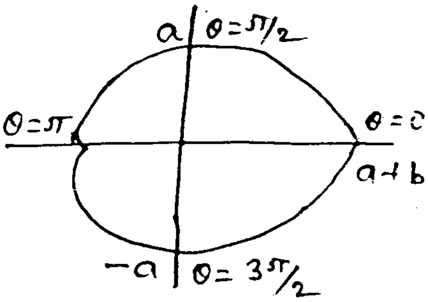
not Jordan curve
Since curve intersect at origin.
closed curve.



Lema-cone

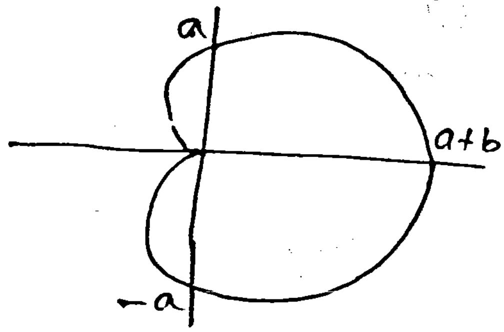
Note:- Jordan arc does not intersect itself.

$a > b$, Cardiod



Jordan curve
and closed curve

$a = b$, Cardiod



Jordan curve
and closed curve.

Note:- Any curve intersect itself not a Jordan arc and not a Jordan curve.

Problem:- $r = 2+5\cos\theta$; $\theta \in [0, 2\pi]$

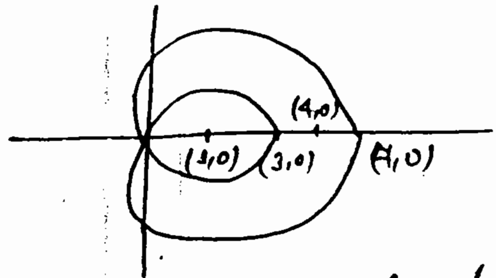
Solution:- Here $a=2$, $b=5$

$a<b \Rightarrow$ lema-cone

not Jordan curve but closed curve.

GATE

Problem:-



$$I_1 = \int_C \frac{1}{z-1}\,dz \quad \text{and} \quad I_2 = \int_C \frac{1}{z-4}\,dz$$

then $\boxed{I_1 = 2I_2}$

$\eta(\gamma, 1) = 2$ where η = winding number

$\eta(\gamma, 4) = 1$ γ = curve

$\eta(\gamma, 8) = 0$ 1, 4, 8, 3 = Points.

$\eta(\gamma, 3)$ = not defined

Smooth curve:- If $\gamma(t) = x(t) + iy(t)$ such that $x(t)$ and $y(t)$ both are differentiable on closed interval $[a,b]$ and their derivatives $x'(t)$, $y'(t)$ does not vanish together then $\gamma(t)$ is referred as <u>smooth curve</u>

Note:- $\gamma(t) = x(t) + iy(t)$

$$\arg[\gamma(t)] = \tan^{-1}\left[\frac{y(t)}{x(t)}\right] = \theta$$

$$\gamma'(t) = x'(t) + iy'(t)$$

$$\arg[\gamma'(t)] = \tan^{-1}\left[\frac{y'(t)}{x'(t)}\right]$$

Now, $y'(t) = \frac{dy}{dt}$, $x'(t) = \frac{dx}{dt}$

$$\frac{y'(t)}{x'(t)} = \frac{dy}{dx} = \tan\alpha$$

$$\arg[\gamma'(t)] = \tan^{-1}(\tan\alpha) = \alpha$$

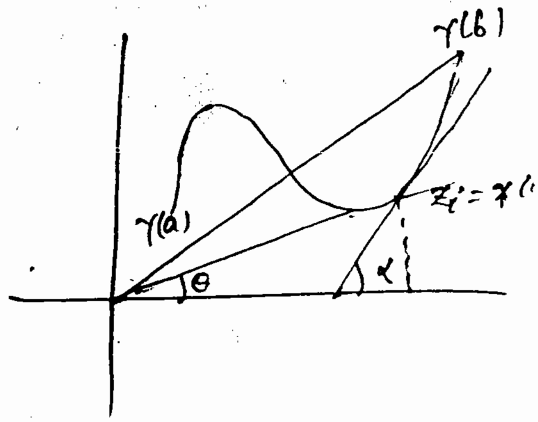
For the curve $\gamma(t)$, $\arg[\gamma'(t)]$ is the angle which tangen to the curve γ makes the +ve direction of x-axis

i.e. $\tan[\arg(\gamma'(t))]$ is slope of tangent to the curve $\gamma(t)$ at t.

Hence we can say $\gamma(t)$ is smooth $\forall\ t\in(a,b)$ if $\gamma'(t)$ exist $\forall\ t\in(a,b)$ and $\gamma'(t)$ is not equal to zero for any $t\in(a,b)$.

$\gamma(t) = x(t) + iy(t)$ is smooth iff $x(t)$ and $y(t)$ are differentiable function of t and derivative does not vanish together.

$P = \{a = t_0, t_1, t_2, \dots t_n = b\}$



$\gamma : [a,b] \to \mathbb{C}$

$\gamma(t) = x(t) + iy(t)\ \forall\ t\in[a,b]$

$\gamma'(t) = x'(t) + iy'(t)$

$z_i = \gamma(t_i)$

$$\theta = \tan^{-1}\left[\frac{y(t_i)}{x(t_i)}\right] = \arg[\gamma(t_i)]$$

$$\alpha = \tan^{-1}\left[\frac{y'(t_i)}{x'(t_i)}\right]$$

→ argument of $\gamma'(t)$ is just the slope of the curve $y = f(x)$.

$$\boxed{\alpha = \arg\gamma'(t_i)}$$

Problem:- $\gamma(t) = e^{it}$; $t \in [0, \pi]$

What is the slope of the tangent to the curve γ at $t = \pi/3$.

Solution:- $\gamma(t) = \cos t + i \sin t$

$\gamma'(t) = -\sin t + i \cos t$

$$\alpha = \arg(\gamma'(t)) = \tan^{-1}\left(\frac{\cos t}{-\sin t}\right) = \tan^{-1}(-\cot \pi/3)$$

$$\alpha = \tan^{-1}\left(-\frac{1}{\sqrt{3}}\right) = -\pi/6 \text{ or } 5\pi/6$$

i.e. $\boxed{\alpha = -\pi/6 \text{ or } 5\pi/6}$

slope $\boxed{\tan\alpha = -\frac{1}{\sqrt{3}}}$ Ans

Piecewise Smooth Curve:- If $\gamma(t)$ is defined on $[a,b]$ and smooth on each sub-interval $[t_{r-1}, t_r]$

Where $[a,b] = \{a = t_0, t_1, t_2, \ldots, t_n = b\}$

Where $t_0, t_1, \ldots t_n$ define the partition of $[a,b]$.

Winding Number (Index of a curve):-

It is always for the closed curve.

Let γ be a closed curve in $\mathbb{C}$ and 'a' is any complex number. The winding number of γ or index of γ w.r.t. 'a' is the number of times γ revolves around 'a' (i.e. winds a). It is denoted by $\eta(\gamma, a)$ and it is always integer.

OR. Let γ be a closed curve in $\mathbb{C}$ s.t. 'a' is lying inside γ then

$\frac{1}{2\pi i} \int_{\gamma} \frac{1}{z-a}\, dz$ is defined as winding number of 'a'.

i.e. $\boxed{\eta(\gamma, a) = \frac{1}{2\pi i} \int_{\gamma} \frac{1}{z-a}\, dz}$ → V. Imp.

e.g.:- Let $\gamma(t) = re^{it}$; $t \in [0, 2k\pi]$.

$\eta(\gamma, 0) = k$ = index of γ

e.g.:- If $\gamma(t) = a + e^{it}$; $t \in [0, 2n\pi]$

Here $z = a + e^{it}$

$dz = ie^{it}\, dt$

$$I = \frac{1}{2\pi i} \int_{\gamma} \frac{1}{z-a}\, dz$$

$$= \frac{1}{2\pi i} \int_{0}^{2n\pi} \frac{i\cancel{e^{it}}}{\cancel{e^{it}}}\, dt = \frac{1}{2\pi} [2n\pi - 0] = n.$$

Where the path is travelled in anticlockwise or counterclockwise direction.

i.e. path is positive.

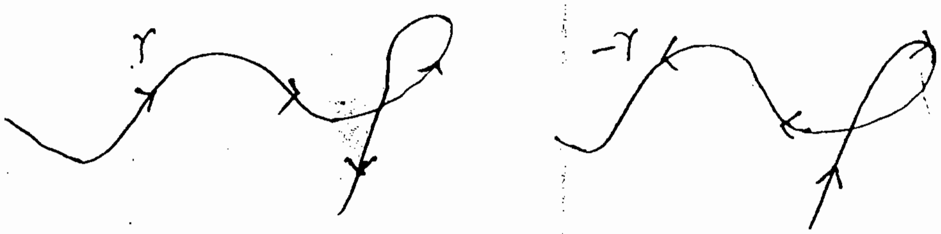
Note:- (i) Whenever we integrate $f(z)$ along a curve then +ve orientation of the integration is considered when the values of $f(z)$ on the points on left of the curve is taken into account.

V. Imp (ii) In case of closed curve when we move anti-clocka direction and value of $f(z)$ inside the clockwise taken then it is called +ve orientation.

Hence if we have a +ve orientation of a integra just taking -ve sign of integral we can have a value of integral. When the points outside the curve are taken account.

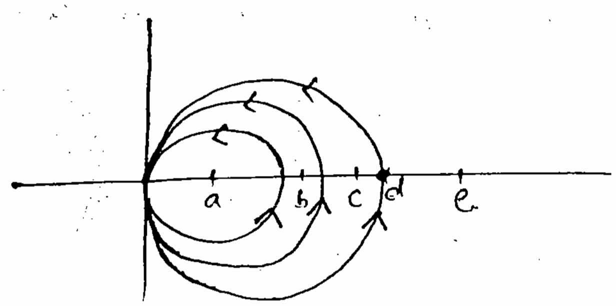
ii) If γ is travelled in anticlockwise direction then index is taken +ve. But in clockwise we take -ve.

Hence $\boxed{\eta(-\gamma, a) = -\eta(\gamma, a)}$.



v) If point 'a' lies outside the curve then index is taken to be 0 i.e. $\eta(\gamma, a) = 0$

i.e. the point which lie outside the closed curve has 0 winding no. for that closed curve.



$\eta(\gamma, a) = 3$

$\eta(\gamma, b) = 2$

$\eta(\gamma, c) = 1$

$\eta(\gamma, d) =$ not defined

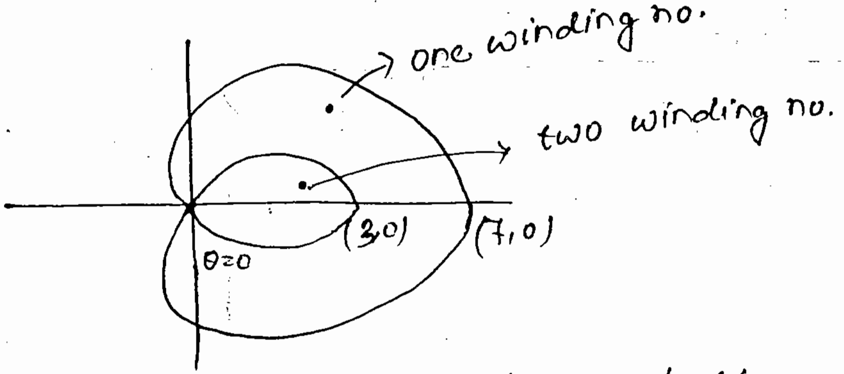
$\eta(\gamma, e) = 0$

Where γ is +ve orientation.

i). The point around which we winds does not lie on the boundary.

i.e. the winding number on any closed curve is not defined w.r.t. all these pts which lie on the curve e.g.:- $r = 2 + 5\cos\theta$; $\theta \in (0, 2\pi)$

When $\theta = 0$, $r = 7$



Any number between $(0,0)$ & $(3,0)$ has winding no. $= 2$.

,, ,, ,, $(3,0)$ & $(7,0)$,, ,, $= 1$.

V. Imp

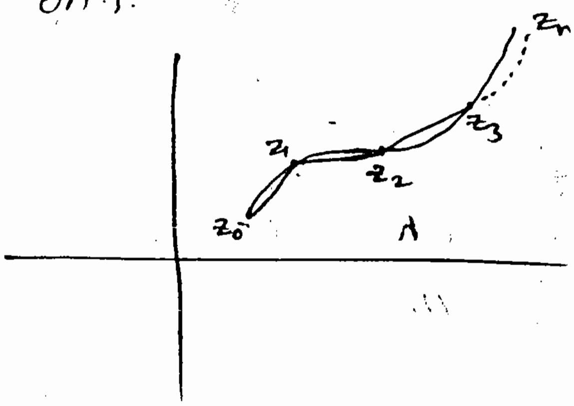
(vi) In multivalued functions, we cannot find winding number. In order to find winding number, functions have to be single valued.

$\longrightarrow$ $\gamma(t) = x(t) + iy(t)$; $t \in [a,b]$

Let $P = \{a = t_0, t_1, t_2, \dots, t_n = b\}$ be a partition of $[a,b]$

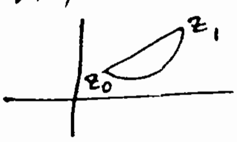
Define $\gamma(t_i) = z_i$

$\Rightarrow$ z_i represents a complex number on γ.

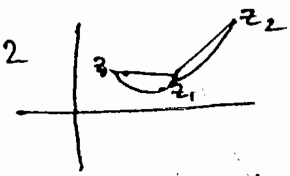


$S_n = \sum_{i=1}^{n} |z_i - z_{i-1}|$ (Real no.)

Let n=1



Let n=2



$\Rightarrow$ S_n is <u>non-decreasing</u>.

$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \sum_{i=1}^{n} |z_i - z_{i-1}| =$ length of the arc provided limit exists.

$|z_i - z_{i-1}| = |x(t_i) + iy(t_i) - x(t_{i-1}) - iy(t_{i-1})|$

$\Rightarrow |z_i - z_{i-1}| = |x(t_i) - x(t_{i-1})| + i|y(t_i) - y(t_{i-1})|$

$= \sqrt{[x(t_i) - x(t_{i-1})]^2 + [y(t_i) - y(t_{i-1})]^2}$

$\geq |x(t_i) - x(t_{i-1})| \qquad [\because \sqrt{x^2+y^2} \geq |x| \text{ or } |y|]$

$\Rightarrow \sum |z_i - z_{i-1}| \geq \sum |x(t_i) - x(t_{i-1})|$

If the arc is rectifiable i.e. (length of the curve can be measured) iff $x(t)$ and $y(t)$ are the functions of bounded variation.

→ $\boxed{\text{Complex Integral}}$:- Let $f(z)$ be a function defined on domain D and γ is the curve lie completely in D.

where $\gamma(t) = x(t) + iy(t)$; $t \in [a,b]$.

further let P be a partition of $[a,b]$

Define the sum

$$S_n = \sum_{r=1}^{n} f(\delta_r) \cdot |z_r - z_{r-1}|$$

where $\delta_r = x(t_r') + iy(t_r')$ & $t_r' \in [t_{r-1}, t_r]$

then $\lim_{n\to\infty} S_n$ exists finitely is called complex line integral of $f(z)$ along γ and written as

$$\lim_{n\to\infty} S_n = \int_{\gamma} f(z)\,dz$$

$$\int_\gamma f(z)\,dz = \int_a^b f(\gamma(t))\,\gamma'(t)\,dt$$

(i) $$\boxed{\int_\gamma f(z)\,dz = \int_a^b f(\gamma(t))\,\gamma'(t)\,dt}$$ → Imp

where
$$\gamma(t) = x(t) + i y(t)$$
$$z(t) = x(t) + i y(t)$$
$$dz = [x'(t) + i y'(t)]\,dt$$
$$dz = \gamma'(t)\,dt.$$

(ii) If $w(t) = u(t) + i v(t)$; t is real $\in [a,b]$

$$\boxed{\int_a^b w(t)\,dt = \int_a^b u(t)\,dt + i\int_a^b v(t)\,dt}$$ → Imp.

(iii)
$$\boxed{\begin{aligned}\int_a^b u(t)\,dt &= \mathrm{Re}\int_a^b w(t)\,dt \\ \int_a^b v(t)\,dt &= \mathrm{Im}\int_a^b w(t)\,dt\end{aligned}}$$

V. Imp

Note:- $\mathrm{Re}\int_\gamma f(z)\,dz \neq \int_\gamma \mathrm{Re}\, f(z)\,dz$

e.g.:- Let $\gamma(t) = it$; $t \in [0,1]$

$f(z) = 1$, $\gamma'(t) = i$

$$\mathrm{Re}\int_\gamma f(z)\,dz = \mathrm{Re}\int_0^1 f[\gamma(t)]\,\gamma'(t)\,dt$$

$$= \mathrm{Re}\int_0^1 1\cdot i\,dt = \mathrm{Re}[i] = 0$$

$$\int_\gamma [\mathrm{Re}\, f(z)]\,dz = \int_\gamma 1\, dz = \int_0^1 \gamma'(t)\,dt = \int_0^1 i\,dt = i$$

i.e. $\mathrm{Re} \int_\gamma f(z)\,dz \neq \int_\gamma \mathrm{Re}(f(z))\,dz$

(iv) $\boxed{\left| \int_a^b \omega(t)\,dt \right| \leq \int_a^b |\omega(t)|\,dt, \quad t \text{ is real}}$ → Imp.

(V) $\int_\gamma f(z)\,dz = \int_\gamma [u(x,y) + i v(x,y)]\,[dx + i dy]$

$$= \int_\gamma (u\,dx - v\,dy) + i\,(v\,dx + u\,dy)$$

i.e. $\boxed{\int_\gamma f(z)\,dz = \int_\gamma (u\,dx - v\,dy) + i(v\,dx + u\,dy)}$ → V. Imp

- If partial derivative of u & v exists and γ is closed u & v satisfy Green function. then by Green's theorem

Imp $\boxed{\int_\gamma f(z)\,dz = \iint_R \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy + i \iint_R \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dx\,dy}$

Where R is a region bdd by γ.

Problem:- Let $f(z) = x^2+y^2+i.2xy$ and $\gamma : y = x^2$

Integrate $f(z)$ between $(0,0)$ and $(1,1)$ along this curve.

Solution:- Here $u = x^2+y^2$, $v = 2xy$

$$y = x^2$$

$$dy = 2x\,dx$$

$$\int_\gamma f(z)dz = \int_\gamma (u\,dx - v\,dy) + i\,(v\,dx + u\,dy)$$

$$= \int_0^1 [(x^2+y^2)\,dx - 2xy\,dy] + i\,[2xy\,dx + (x^2+y^2)dy]$$

$$= \int_0^1 [(x^2+x^4)dx - 2x\cdot x^2\cdot 2x\,dx] + i\,[2x\cdot x^2 dx + (x^2+x^4$$

$$= \left[\frac{x^3}{3} + \frac{x^5}{5} - 4\cdot\frac{x^5}{5}\right]_0^1 + i\left[\frac{2x^3}{4} + 2\frac{x^4}{4} + \frac{2x^6}{6}\right]_0$$

$$= \frac{1}{3} + \frac{1}{5} - \frac{4}{5} + i\left[\frac{2}{4} + \frac{2}{4} + \frac{2}{6}\right]$$

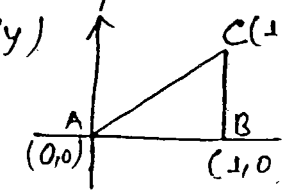
$$= -\frac{4}{15} + \frac{4}{3}i \quad \underline{\text{Ans}}$$

Problem:- $f(z) = (x^2+y^2) + i.2xy$ and $\gamma = AB + BC$

AB : on x-axis from $(0,0)$ to $(1,0)$

BC : on y-axis from $(1,0)$ to $(1,1)$

Solution:- $\int_\gamma f(z)\,dz = \int_\gamma (u\,dx - v\,dy) + i\,(v\,dx + u\,dy)$



Now. $\int_\gamma (u\,dx - v\,dy) = \int_{AB} (u\,dx - v\,dy) + \int_{BC} (u\,dx - v\,dy)$

$$= \int_{AB} (x^2+y^2)dx - 0 + \int_{BC} 0 - 2xy\,dy$$

along AB $y=0 \therefore dy=0$ along BC, $x=1$

$$= \int_0^1 x^2 dx + \int_0^1 -2y \cdot 1\, dy$$

$$= \left[\frac{x^3}{3}\right]_0^1 - 2\left[\frac{y^2}{2}\right]_0^1 = \frac{1}{3} - 1 = -\frac{2}{3}$$

Again $\int_\gamma v\,dx + u\,dy = \int_{AB} v\,dx + u\,dy + \int_{BC} v\,dx + u\,dy$

$$= \int_{AB} 2xy\,dx + (x^2+y^2)dy + \int_{BC} 2xy\,dx + (x^2+y^2)dy$$

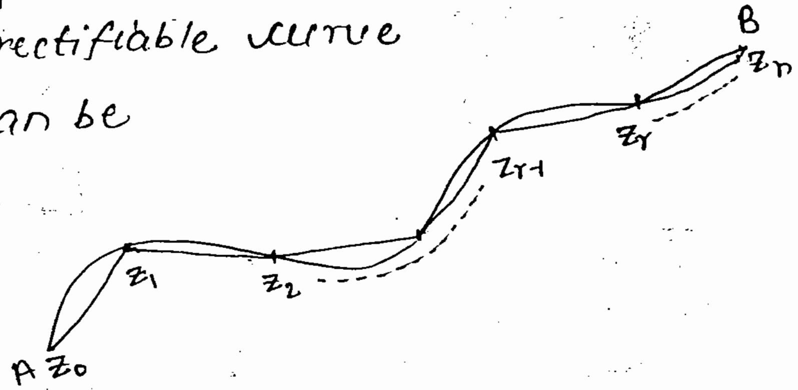
along AB $y=0 \Rightarrow dy=0$ along BC, $x=1 \Rightarrow dx=0$

$$= \int_0^1 0 + x^2 \cdot 0 + \int_0^1 0 + (1+y^2)dy$$

$$= \left[y + \frac{y^3}{3}\right]_0^1 = 1 + \frac{1}{3} = \frac{4}{3}$$

i.e. $\boxed{\int_\gamma f(z)\,dz = -\frac{2}{3} + \frac{4}{3}i}$ Ans

$\boxed{\text{Rectifiable Curve}}$:- A curve between the point A and B is said to be rectifiable curve iff its length can be measured finitely.



i.e $\lim_{n\to\infty} \sum_{r=1}^{n} |z_r - z_{r-1}|$ exists finitely $=$ Arc length AB

$$= \int_{z_0}^{z_n} |dz| = \int_{t_0}^{t_n} |\gamma'(t)|\,dt$$

$$= \int_{t_0}^{t_n} |x'(t) + iy'(t)|\,dt = \int_{t_0}^{t_n} \sqrt{[x'(t)]^2 + [y'(t)]^2}\,dt$$

Some Important Results :-

1). $\int_{z_0}^{z_1} |dz| =$ Arc length joining z_0 to z_1

2). $\int_{z_0}^{z_1} dz = z_1 - z_0$

3). $\int_C \frac{1}{z-a} dz = 2\pi i$

Proof :- $C : |z-a| = r$

$z = a + re^{it} \quad ; t \in [0, 2\pi]$.

$dz = rie^{it} dt$

$$\int_C \frac{1}{z-a} dz = \int_0^{2\pi} \frac{1}{\cancel{re^{it}}} \cdot \cancel{r} i \cancel{e^{it}} dt = i \int_0^{2\pi} dt = 2\pi i$$

4). $$\int_C (z-a)^n dz = \begin{cases} 0 & ; n \neq -1 \\ 2\pi i & ; n = -1 \end{cases}$$

$C : \ z = a + re^{it} \quad ; t \in [0, 2\pi]$.

5). **Negative of γ** :- If $\gamma(t)$ gives curve from $\gamma(a)$ to $\gamma(b)$ then $\gamma(-t)$ gives the same curve but travels from $\gamma(b)$ to $\gamma(a)$.

and $\int_\gamma f(z) dz = -\int_{-\gamma} f(\gamma(-t))\gamma'(-t) dt$

$$\boxed{\int_\gamma f(z) dz = -\int_{-\gamma} f(z) dz} \rightarrow \text{Imp}$$

ML- Inequality :-

$$\left| \int_{\gamma} f(z)dz \right| \le ML$$

Where $f(z)$ is defined on a domain D and γ is a curve completely lie in D and $|f(z)| \le M \ \forall \ z \in D$

ML- Inequality is standard estimation of the complex integral or estimate the integral without actual evaluation.

OR, $f(z)$ is bounded on D and γ lies in D

$$\left| \int_{\gamma} f(z)dz \right| \le \int_{\gamma} |f(z)|\,|dz|$$

$$\le M \int_{\gamma} |dz|$$

$$\boxed{\left| \int_{\gamma} f(z)dz \right| \le M\ell}$$

Where 'ℓ' is length and γ is rectifiable curve. This inequality is called M-ℓ- inequality and gives you estimation of complex integration.

Problem: $\displaystyle\int_{\gamma} \frac{(z^2+3)\, e^{iz} \log z}{z^2-2}\, dz$ $C: \{z : z = 2e^{i\theta},\ 0 \le \theta \le \pi/3$
$|z| = 2.$

Solution:- Circumference of circle $= 2\pi r$

i.e $\ell = \frac{2\pi r}{6} = \frac{2\pi \cdot 2}{6} = \frac{2\pi}{3}$ $\left[\because \frac{2\pi}{C} = \frac{\pi}{3} \right]$

i.e. $\ell = \frac{2\pi}{3}$.

$$\left|\int_C \frac{(z^2+3)\, e^{iz} \log z}{(z^2-2)}\, dz\right| \leq \int_C \frac{|z^2+3|\,|e^{iz}|\,|\log z|}{|z^2-2|}\, |dz|$$

$$\leq \int_C \frac{(|z|^2+3)\, e^{-y}\, |\log r + i\theta|}{(|z|^2-2)}\, |dz|$$

$$e^{iz} = e^{i(x+iy)} = e^{ix-y}$$

$$|e^{iz}| = e^{-y}$$

$$e^{-y} < 1$$

$$\leq \int_C \frac{(4+3)\, e^{-y}\, |\log 2 + i\frac{\pi}{3}|}{(4-2)}\, |dz|$$

$$\leq \int_C \frac{7e^{-y}\left(|\log 2| + |i|\, \frac{\pi}{3}\right)}{2}\, |dz|$$

$$\leq \frac{7e^{-y}\left(\log 2 + \frac{\pi}{3}\right)}{2} \qquad \int_C |dz| = l = \frac{2\pi}{3}$$

$$\leq \frac{7e^{-y}\,(3\log 2 + \pi)}{\cancel{2}\cdot 3} \cdot \frac{\cancel{2}\pi}{3}$$

$$\leq \frac{7e^{-y}\,\pi\,(3\log 2 + \pi)}{9} \qquad \boxed{e^{-y} < 1}$$

$$\leq \frac{7\pi}{9}\,(3\log 2 + \pi)$$

$$\therefore \boxed{\left|\int_\gamma \frac{(z^2+3)\, e^{iz} \log z}{z^2-2}\, dz\right| \leq \frac{7\pi}{9}\,(3\log 2 + \pi)} \quad \underline{\text{Ans}}$$

Result :- If $f(z)$ is continuous in a domain D then the following statements are equivalent :-

(i) $f(z)$ has anti-derivative i.e. it has primitive i.e. it has indefinite integral (say F in D)

i.e. $F'(z) = f(z) \Rightarrow F(z) = \int f(z)dz$

(ii) The integral of $f(z)$ along any path lying between any two points in D is independent of path i.e. the integral depends only in terminal points.

(iii) The integral of $f(z)$ along any closed contour C is zero.

Proof :- $\boxed{(i) \Rightarrow (ii)}$

$$\int_\gamma f(z)dz \quad ; \quad t \in [a,b]$$

$$\gamma(a) = z_0 \quad \gamma(b) = z_n$$

$$F'(z) = f(z)$$

$$f(z) = f(\gamma(t))$$

$$\int_\gamma f(z)dz = \int_\gamma f(\gamma(t))dz = \int_a^b f(\gamma(t))\gamma'(t)dt$$

$$= [F(\gamma(t))]_a^b$$

$$= F(\gamma(b)) - F(\gamma(a))$$

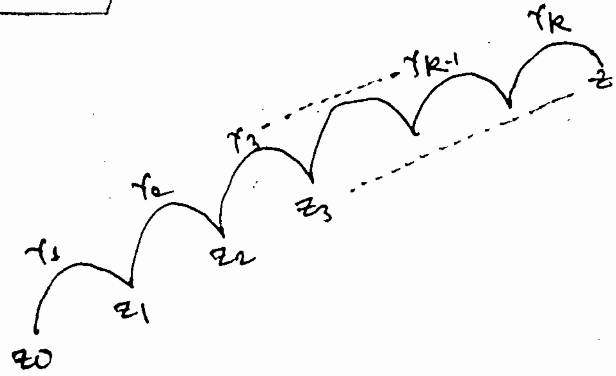
$$= F(z_n) - F(z_0)$$

$\boxed{(ii) \Rightarrow (iii)}$

$\gamma'(t) \Rightarrow \gamma$ is smooth between the point $\gamma(a)$ and $\gamma(b)$.

If curve γ is not smooth but piecewise smooth then

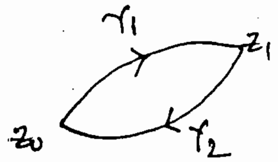
$$\gamma = \gamma_1 + \gamma_2 + \cdots + \gamma_k$$



then $\int_\gamma f(z)dz = \sum_{i=1}^{k} \int_{\gamma_i} f(z)dz$

$$= \sum_{i=1}^{k} F(z_i) - F(z_{i-1})$$

$$= F(z_k) - F(z_0)$$



If closed curve

$$\gamma = \gamma_1 - \gamma_2$$

$$\int_{\gamma_1} f(z)dz = \int_{\gamma_2} f(z)dz$$

$$\Rightarrow \int_{\gamma_1} f(z)dz + \int_{-\gamma_2} f(z)dz = 0$$

Remember

$$\boxed{\int_{-\gamma} f(z)\,dz = -\int_{\gamma} f(z)dz}$$

$$\Rightarrow \int_{\gamma_1-\gamma_2} f(z)dz = 0$$

$\boxed{(iii) \Rightarrow (i)}$

$$F(z) = \int_a^z f(t)dt$$

$$F(z+h) = \int_a^{z+h} f(t)dt$$

Now, $\dfrac{F(z+h) - F(z)}{h} = \dfrac{1}{h}\int_z^{z+h} f(t)dt$

Again $\dfrac{F(z+h) - F(z)}{h} - f(z) = \dfrac{1}{h}\int_z^{z+h} f(t)dt - f(z)$

or, $\left|\dfrac{F(z+h) - F(z)}{h} - f(z)\right| = \left|\dfrac{1}{h}\int_z^{z+h} [f(t) - f(z)]dt\right|$

$$\leq \frac{1}{|h|} \int_{z}^{z+h} |f(t) - f(z)|\,|dt|$$

$$\leq \frac{\varepsilon}{\cancel{|h|}} \cdot \cancel{|h|}$$

$$< \varepsilon$$

$$\Rightarrow \lim_{h \to 0} \left[\frac{F(z+h) - F(z)}{h} \right] = f(z)$$

$$\Rightarrow \boxed{F'(z) = f(z)} \quad \text{Hence proved}$$

Note:- $\int_{|z|=1} f(z)\,dz = 0$ iff 0 is only singularity and $f(-z) = f(z)$

Simply Connected Region:- Let D be a region such that any closed contour lying completely in D can be shrunk to a point with leaving the region D then D is said to be simply connected otherwise multiply connected.

Cauchy Theorem:- If $f(z)$ is analytic in D when D is any open region and $f'(z)$ is continuous then integral of $f(z)$ along any closed contour lying in D is zero.

Proof:- $f'(z)$ is continuous

$$f'(z) = u_x + i v_x$$

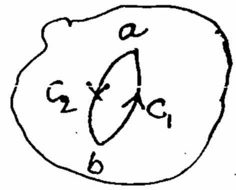
By Green's theorem

$$\int_C f(z)\,dz = \iint_A \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) dx\,dy + i \iint \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right) dx\,dy$$

$= 0$ using C-R equⁿ.

Note :- If integral of $f(z)$ is zero along any closed curve inside D then it imply the integral is path independent

Proof :- $\gamma = C_1 + (-C_2)$



$$\int_{\gamma} f(z)dz = 0$$

$$\Rightarrow \int_{C_1+(-C_2)} f(z)dz = 0$$

$$\Rightarrow \int_{C_1} f(z)dz + \int_{-C_2} f(z)dz = 0$$

$$\Rightarrow \int_{C_1} f(z)dz - \int_{C_2} f(z)dz = 0$$

$$\Rightarrow \boxed{\int_{C_1} f(z)dz = \int_{C_2} f(z)dz}$$

Cauchy - Goursat Theorem :- If $f(z)$ is analytic in D then integral of $f(z)$ along any closed curve lying in D is zero.

i.e Goursat slacks the condition of quantity of $f'(z)$.

V.Imp

Result :- If $f(z)$ is analytic in D except at $z_0 \in D$ but $f(z)$ is continuous in D then integral of $f(z)$ along any closed curve lying in D is zero.

M. Imp (Subjective)

[Cauchy Integral Formula] :- If $f(z)$ is analytic in D and $a \in D$ then $\boxed{f(a) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{z-a} dz}$

[Proof] :- $\gamma : |z-a| = r$

$$z = a + re^{i\theta}$$

$$\theta \in [0, 2\pi]$$

Define
$$\phi(z) = \begin{cases} \frac{f(z)-f(a)}{z-a} & ; z \neq a \\ f'(a) & ; z = a \end{cases}$$

$$\lim_{z \to a} \phi(z) = \lim_{z \to a} \frac{f(z)-f(a)}{z-a}$$

$$\Rightarrow \lim_{z \to a} \phi(z) = f'(a)$$

$$\Rightarrow \lim_{z \to a} \phi(z) = \phi(a)$$

Hence $\phi(z)$ is analytic in $D - \{a\}$ and continuous at $z = a$.

Hence $\int_{\gamma} \phi(z) dz = 0$ $\qquad \gamma : |z-a| = r$

$$\Rightarrow \int_{\gamma} \frac{f(z)-f(a)}{z-a} dz = 0$$

$$\Rightarrow \int_{\gamma} \frac{f(z)}{z-a} dz - \int_{\gamma} \frac{f(a)}{z-a} dz = 0$$

$$\Rightarrow f(a) \int_{\gamma} \frac{1}{z-a} dz = \int_{\gamma} \frac{f(z)}{z-a} dz$$

$$\Rightarrow \boxed{f(a) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z-a}\, dz} \qquad \left[\because \int_\gamma \frac{1}{z-a}\, dz = 2\pi i \ ; \ \gamma : |z\right.$$

Cauchy integral formula gives the value of function a a point in terms of nbd of that points.

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(t)}{(t-z)}\, dt$$

$$f'(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(t)}{(t-z)^2}\, dt$$

$$\Rightarrow \boxed{f^n(z) = \frac{n!}{2\pi i} \int_\gamma \frac{f(t)}{(t-z)^{n+1}}\, dt}$$

$$\Rightarrow \boxed{f^n(a) = \frac{n!}{2\pi i} \int_\gamma \frac{f(z)}{(z-a)^{n+1}}\, dz}$$

Note :- As for any $z \in D$, $f^n(z)$ exists for $\forall\ n \in \mathbb{N}$. We can say the derivative of analytic function is analytic.

Hence " If a function has primitive in D then the function is analytic in D"

Morera Theorem :- If $f(z)$ is continuous in D and integral of $f(z)$ along any closed contour lying is zero then $f(z)$ is analytic. [Converse of Cauchy Theorem].

Application of Cauchy Integral Formula :-

Cauchy Inequality :- If $f(z)$ is analytic within a circle C given by $|z-a|=R$ and if $|f(z)| \le M$ on C then $\boxed{|f^n(a)| \le \frac{M n!}{R^n}}$

$C : |z-a|=R$

Now, $$f^n(a) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-a)^{n+1}}\,dz$$

$$|f^n(a)| = \left|\frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-a)^{n+1}}\,dz\right|$$

$$\le \frac{n!}{2\pi}\int_C \frac{|f(z)|}{|z-a|^{n+1}}\,|dz|$$

$$\le \frac{n!}{2\pi} M \int_C \frac{|dz|}{R^{n+1}}$$

$$\le \frac{n!\,M}{2\pi R^{n+1}}\int_C |dz|$$

i.e. $$|f^n(a)| \le \frac{n!\,M}{\cancel{2\pi} R^{n+1}}\cdot \cancel{2\pi} R = \frac{n!\,M}{R^n}$$

Hence $$\boxed{|f^n(a)| \le \frac{n!\,M}{R^n}}$$ → which is Cauchy Inequality.

Entire function :- A function which is analytic in the z-plane is called an entire function.

(c-plane or excluded the point of ∞).

Entire means have only ∞ singularity.

V.V. Imp.

$\boxed{C_6}$:- **Liouville's Theorem** :- If $f(z)$ is entire and bounded function then $f(z)$ is constant.

e.g. (i) $f(z) = \sum_{i=0}^{n} a_i z^i = P(z) \longrightarrow$ unbdd

(ii) $f(z) = e^z \longrightarrow$ unbdd

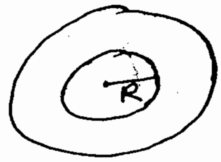
(iii) $f(z) = \sin z, \cos z \longrightarrow$ unbdd

(iv) $f(z) = \dfrac{P(z)}{e^z}, \quad \dfrac{\sin z}{e^z}$

V. Imp (v) If $f(z)$ is entire function then $e^{f(z)}$ is also entire function.

Proof :- First we prove Cauchy inequality

Now, $|f^n(z)| \leq \dfrac{n!\,M}{R^n}$



Now. $n=1$, $|f'(z)| \leq \dfrac{1!\,M}{R} = \dfrac{M}{R}$

As $f(z)$ is entire then R can be taken as large as possible

i.e. $R \to \infty$

$\Rightarrow |f'(z)| \leq \dfrac{M}{R} \to 0$ as $R \to \infty$

i.e. $|f'(z)| = 0 \ \forall z$

$\Rightarrow f(z)$ is constant.

Hence proved.

Note: (i) If $f(z)$ is not constant then either $f(z)$ is not entire or not bdd (ubdd)

(ii) If $f(z)$ and $g(z)$ are differentiable at $z=a$ then $(f \circ g)(z)$ and $(g \circ f)(z)$ are also differentiable at $z=a$.

→ (a) not constant ⇒ not entire or not bounded.

(b) not constant and bdd ⇒ not entire

(c) not constant and entire ⇒ not bdd

(d) entire ⇒ constant or unbdd

(e) bdd ⇒ constant or not entire.

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G7:- **Theorem**:- If $f(z)$ is entire and real part of $f(z)$ is bounded then $f(z)$ is constant.

Proof:- Given $f(z) = u + iv$ is entire

and $|u| \leq M$

Now, construct $g(z) = e^{f(z)} \rightarrow$ entire

$$g(z) = e^{u+iv} = e^{u} \cdot e^{iv}$$

$\Rightarrow |g(z)| = e^{u} < e^{M}$ [exponential function is increasing]

$\Rightarrow$ $g(z)$ is bdd

Hence $g(z)$ is entire and bounded function then $g(z)$ is constant function.

i.e. $e^{f(z)} = A$; $A \in \mathbb{C}$

$f(z) = \log A$

i.e. $f(z)$ is also constant. Hence proved

$\boxed{C_8}$:- **Theorem** :- If $f(z)$ is entire function such that imaginary part of $f(z)$ is non-negative then $f(z)$ is constant.

Proof :- Given $f(z) = u + iv \longrightarrow$ Entire function

Also given $v \geq 0$

Construct

$$g(z) = e^{if(z)} \longrightarrow \text{entire function}$$

Imp

$$\Rightarrow g(z) = e^{i(u+iv)}$$

$$\Rightarrow g(z) = e^{iu-v} = e^{iu} \cdot e^{-v}$$

$$\Rightarrow |g(z)| = e^{-v} \cdot 1 \leq 1$$

$$\Rightarrow |g(z)| \leq 1$$

$$\boxed{\begin{array}{l} v \geq 0 \\ e^{v} \geq e^{0} \\ \frac{1}{e^{v}} \leq 1 \end{array}}$$

Hence $g(z)$ is bounded.

i.e. $g(z)$ is entire and bounded function

Hence by Liouville's theorem $g(z)$ is constant.

say $g(z) = A$

$$e^{if(z)} = A$$

$$\Rightarrow if(z) = \log A$$

$$\Rightarrow f(z) = -i \log A$$

Hence $f(z)$ is constant.

Hence proved.

$\boxed{C_9}$: $\boxed{\text{Theorem}}$: If $f(z) = u+iv$ is entire function such that $au+bv \le c$, where $a, b, c \in \mathbb{R}$ then $f(z)$ is constant.

$\boxed{\text{Proof}}$:- $f(z) = u+iv \rightarrow$ Entire function

$au+bv \le c$ where $a, b, c \in \mathbb{R}$

Construct $g(z) = e^{(a-ib)f(z)} \rightarrow$ entire function

$\underline{\text{Imp}}$

$$= e^{(a-ib)(u+iv)}$$

$$= e^{au+bv+i(av-bu)}$$

$$= e^{au+bv} \cdot e^{i(av-bu)}$$

$$\Rightarrow |g(z)| = e^{au+bv} \cdot 1.$$

since $au+bv \le c$

$e^{au+bv} \le e^c$ [since exponential function is increasing function]

$$\Rightarrow |g(z)| \le e^c$$

i.e. $g(z)$ is bounded

Hence $g(z)$ is entire and bounded function. then by Liouville's theorem $g(z)$ is constant.

say $g(z) = A$

$e^{(a-ib)f(z)} = A \Rightarrow (a-ib)f(z) = \log A$

$$\Rightarrow (a^2+b^2)f(z) = (a+ib)\log A$$

$$\Rightarrow f(z) = \frac{a+ib}{a^2+b^2}\log A$$

i.e. $f(z)$ is constant. Hence proved

C10 **Theorem**:- If $f(z)=u+iv$ is entire function such that $au+bv \geq c$ where $a,b,c \in \mathbb{R}$ then $f(z)$ is constant.

Proof:- Given $f(z)=u+iv \rightarrow$ Entire function

$au+bv \geq c$ where $a,b,c \in \mathbb{R}$

Construct $g(z) = e^{(-a+ib)f(z)} \rightarrow$ Entire function

$$= e^{(-a+ib)(u+iv)}$$

$$= e^{-au-bv} \cdot e^{i(bu-av)}$$

$$\Rightarrow |g(z)| = e^{-au-bv} = \frac{1}{e^{au+bv}}$$

Since $au+bv \geq c$

$$\Rightarrow e^{au+bv} \geq e^{c}$$

$$\Rightarrow \frac{1}{e^{au+bv}} \leq \frac{1}{e^{c}}$$

$$|g(z)| \leq \frac{1}{e^{c}}$$

i.e. $g(z)$ is bounded.

Hence $g(z)$ is entire and bounded function.

Then by Liouville's theorem $g(z)$ is constant.

say $g(z)=A$

$$e^{(-a+ib)f(z)} = A$$

$$\Rightarrow (-a+ib)f(z) = \log A$$

$$\Rightarrow -(a-ib)(a+ib)f(z) = (a+ib)\cdot \log A$$

$$\Rightarrow f(z) = -\frac{(a+ib)}{a^2+b^2}\log A$$

$\Rightarrow f(z)$ is constant. Hence proved.

$\boxed{C_{11}}$: $\boxed{\text{Theorem}}$:- Any analytic function from $\mathbb{C}$ to D where D is bounded is constant [Another form of Liouville's Thm]

GATE

$\boxed{\text{e.g}}$:- $f: \mathbb{C} \to D$; $|z|<1$

If $f(1+i) = 10$ then $f(1-i) = ?$

Since by above theorem C_{10}, function is constant.

Then $f(1-i) = \underline{\underline{10 \text{ (also)}}}$

V.V Imp

$\boxed{C_{12}}$: $\boxed{\text{Theorem}}$:- If $f(z)$ is entire and it has two L.I periods over $\mathbb{R}$ then $f(z)$ is constant.

$\boxed{\text{OR}}$, If $f(z)$ is entire and $\exists$ non-zero z_1 and $z_2 \in \mathbb{C}$ s.t. $f(z+z_1) = f(z+z_2) = f(z)$ $\forall$ $z \in \mathbb{C}$

and $\dfrac{z_1}{z_2}$ is not real

then $f(z)$ is constant

$\boxed{\text{OR}}$, If $f(z)$ is entire function s.t. $f(z+1) = f(z+i) = f(z)$ then $f(z)$ is constant.

$\boxed{\text{Proof}}$: As $1(z_1)$ and $i(z_2)$ for basis of $\mathbb{C}$ over $\mathbb{R}$

then for any $z \in \mathbb{C}$, $\exists$ $a, b \in \mathbb{R}$

s.t. $z = a.1 + b.i$

$$\boxed{z = a.z_1 + b.z_2}$$

$f(z+1) = f(z)$

$f(z+i) = f(z)$

$$\boxed{\begin{array}{l} f(z+z_1) = f(z) \\ f(z+z_2) = f(z) \end{array}}$$

$\Rightarrow$ $f(z+n.1) = f(z+n.i) = f(z)$ $\forall$ $n \in \mathbb{C}$

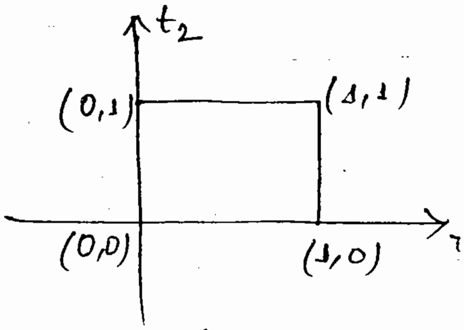
Let $z \in \mathbb{C}$; $z = a+ib = a.1 + b.i$; $a, b \in \mathbb{R}$

$f(z) = f(a.1 + b.i)$

$a = n_1 + t_1$ and $b = n_2 + t_2$ where $n_1, n_2 \in \mathbb{Z}$

$t_1, t_2 \in [0,1)$.

$$\Rightarrow f(z) = f[n_1 + t_1 + i(n_2 + t_2)]$$
$$= f[(n_1 + in_2) + (t_1 + it_2)]$$
$$= f(t_1 + it_2)$$



Since t_1, t_2 lie in $[0,1)$, $f(z)$ takes only those value which are obtained by choosing z from unit square.

Since $f(z)$ is entire and bounded in the unit square.

Hence $f(z)$ is bounded on $\mathbb{C}$.

Then by Liouville's theorem, $f(z)$ is constant.

Hence proved.

$\boxed{\text{Note}}$:- (1) If $f(z)$ is analytic in D then it is bounded on any closed subset of D.

(2) If $f(z)$ is analytic in D and its range can be obtained on $S \subset D$ then bound of $f(z)$ on S is bound of $f(z)$ on any set containing S.

$\boxed{C_{13}}$: **Theorem**:- If $f(z)$ is entire and $|f(z)| \geq 1$ then $f(z)$ is constant.

Proof:- Construct $g(z) = \frac{1}{f(z)} \rightarrow$ entire

$$|g(z)| = \left|\frac{1}{f(z)}\right| = \frac{1}{|f(z)|} \leq 1$$

$$\Rightarrow |g(z)| \leq 1$$

Hence $g(z)$ is bounded and entire function

i.e. By Liouville's theorem $g(z)$ is a constant function

Say $g(z) = A \quad ; A \in \mathbb{C}$

$$\frac{1}{f(z)} = A$$

$$\Rightarrow f(z) = \frac{1}{A} \quad ; A \neq 0$$

$\Rightarrow f(z)$ is constant. Hence proved

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$\boxed{C_{14}}$:- **Theorem**:- If $f(z)$ is entire function satisfied the condition $|f'(z) \cdot f(z)| \leq 1$ then show that $f(z)$ is constant.

Proof:- Construct $g(z) = \frac{1}{2}[f(z)]^2 \longrightarrow$ Entire function.

$\Rightarrow g'(z) = f'(z) \cdot f(z) \rightarrow$ Entire function

$\Rightarrow |g'(z)| = |f'(z) \cdot f(z)| \leq 1$

$\Rightarrow |g'(z)| \leq 1$

Hence $g'(z)$ is bdd and entire then by Liouville's theorem $g'(z)$ is constant.

Say $g'(z) = A$

$\Rightarrow g(z) = Az + B$

$\Rightarrow \frac{1}{2}[f(z)]^2 = Az + B$

$\Rightarrow f(z) = \sqrt{2Az + 2B}$ $\quad$ $z \to \infty \Rightarrow f(z) \to \infty$ so $\boxed{A=0}$

[But $f(z)$ is entire. So $\sqrt{2Az+2B}$ have no zeros then A must be zero because every polynomial has zeros i.e. $A=0$]

$\Rightarrow f(z) = \sqrt{2B}$

Hence $f(z)$ is constant.

$\boxed{\text{Problem}}$:- If u is harmonic and $|u| \le M$ on $\mathbb{R}^2$ then u is constant.

$\boxed{\text{Proof}}$:- Since u is harmonic. Then $\exists$ v s.t. $f(z) = u + iv$ is analytic & $|u| \le M$

$\Rightarrow f(z)$ is constant $\quad$ [$\because$ Re $f(z)$ is bdd]

$\Rightarrow u + iv$ is constant

$\Rightarrow u$ is constant.

NET-DEC 16.

Very Imp.

Result: If $f(z)$ and $g(z)$ are entire satisfying the condition $|f(z)| < |g(z)|$ $\forall z \in \mathbb{C}$
then $f(z) = c \cdot g(z)$ for some complex constant c.

OR. Let V be the vector space of all complex valued function over the field of complex number and $f(z)$ & $g(z)$ satisfies the condition $|f(z)| < |g(z)|$ $\forall z \in \mathbb{C}$
Then the set $S = \{f(z), g(z)\}$ are L.D.

Proof:- Define $h(z) = \dfrac{f(z)}{g(z)}$; $g(z) \neq 0$

$g(z)$ has no-zeros in $\mathbb{C}$.

Then $h(z)$ is entire function as $f(z)$ and $g(z)$ are entire functions.

$$\Rightarrow \quad |h(z)| = \left|\frac{f(z)}{g(z)}\right| = \frac{|f(z)|}{|g(z)|} < 1$$

Hence $|h(z)| \leq 1$

Hence $h(z)$ is bdd.

i.e. $h(z)$ is entire & bdd function.

Then by Liouville's theorem $h(z)$ is a constant function.

Say $h(z) = c$; where c is complex constant

$$\frac{f(z)}{g(z)} = c$$

$\Rightarrow$ $f(z) = c\, g(z)$ for some complex constant c.

Hence Proved

Note:- If $f(x)$ is periodic function with period T then period of $f(ax)$ is $\dfrac{T}{|a|}$.

Problem :- $f(z)$ is analytic within & on $C : |z-2| = 3$. Suppose $|f(z)|$ has maximum value 2 on C. Then an upper bound of $|f^4(2)| = ?$

Solution :- We have $|f^n(a)| \leq \dfrac{M \cdot n!}{R^n}$

For $n = 4$ and $\underset{M}{a} = 2$, $R = 3$.

$$|f^4(2)| \leq \frac{2 \cdot 4!}{3^4}$$

$$= \frac{2 \cdot \cancel{24}^{8}}{9 \cdot \cancel{9}3} = \frac{16}{27}$$

$$\Rightarrow |f^4(2)| \leq \frac{16}{27}.$$

Fundamental Theorem Of Algebra :- Every non-cons. polynomial over $\mathbb{C}$ has at least one zero in $\mathbb{C}$. Hence exactly n zeros in $\mathbb{C}$. where n is the degree of polynomial.

Proof :- $p(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$

$a_n \neq 0$ and $\underline{a_i \in \mathbb{C}}$ be a polynomial of degree n. $z = Re^{i\theta}$

Whereas $z \to \infty$ then $p(z) \to \infty$

Construct $q(z) = \dfrac{1}{p(z)}$.

Let $p(z)$ has no zeros in $\mathbb{C}$.

then $|q(z)| = \dfrac{1}{|p(z)|}$ as $z \to \infty$ $(R \to \infty)$ $|p(z)| \to \infty$

Hence $\frac{1}{|p(z)|}$ can be made as small as we please can be choose R is sufficiently large.

$$\Rightarrow \quad \frac{1}{|p(z)|} < \varepsilon \quad ; \quad R > \delta$$

Hence $q(z)$ is bounded as well as entire.

$\Rightarrow$ $q(z)$ is constant

$\Rightarrow$ $\frac{1}{p(z)}$ is also constant.

$\Rightarrow$ $p(z)$ is also constant, which is contradiction.

Hence $p(z)$ has atleast one zero in $\mathbb{C}$ say α_1

Hence $p(z) = (z-\alpha_1)\, p_1(z)$

where degree of $p_1(z) = n-1$

Apply the same logic, we can prove $p_1(z)$ has one zero α_2 (say)

then $p_1(z) = (z-\alpha_2)\, p_2(z)$

$$\Rightarrow p(z) = (z-\alpha_1)(z-\alpha_2)\, p_2(z)$$

Since degree of $p(z)$ is finite (i.e. n) the process is terminates n steps gives the n zeros.

Hence proved.

$\Rightarrow$ $\boxed{\text{Gauss Theorem}}$:- Let $p(z)$ be a polynomial of degree $n \geq 1$ then every zero of $p'(z)$ also lies in the convex hull of the set of zeros of $p(z)$.

2). **Luca Theorem** :- If all the zeros of the polynomial $p(z)$ lies in the half plane and then zeros of derivative of $p(z)$ also lies in the same half plane.

Proof of (I) :- $p(z) = a_n(z-\alpha_1)(z-\alpha_2)\text{-----}(z-\alpha_n)$

$$p(z) = a_n \prod_{i=1}^{n} (z-\alpha_i) \qquad \text{where } \alpha_k = a_k + ib_k$$

Taking log and then differentiating on both sides, we get

$$\frac{p'(z)}{p(z)} = \prod_{i=1}^{n} \frac{1}{z-\alpha_i}$$

Let $p'(\beta) = 0$ where $\beta = a+ib$.

$$\Rightarrow \sum \frac{1}{\beta-\alpha_i} = 0 \Rightarrow \sum \frac{\bar{\beta}-\bar{\alpha}_i}{|\beta-\alpha_i|^2} = 0$$

$$\Rightarrow \frac{\bar{\beta}-\bar{\alpha}_1}{|\beta-\alpha_1|^2} + \frac{\bar{\beta}-\bar{\alpha}_2}{|\beta-\alpha_2|^2} + \text{----} + \frac{\bar{\beta}-\bar{\alpha}_n}{|\beta-\alpha_n|^2} = 0$$

$$\Rightarrow \bar{\beta} \sum_{i=1}^{n} \frac{1}{|\beta-\alpha_i|^2} = \sum_{i=1}^{n} \frac{\bar{\alpha}_i}{|\beta-\alpha_i|^2}$$

$$\text{Let } \lambda = \sum_{i=1}^{n} \frac{1}{|\beta-\lambda_i|^2}$$

$$\Rightarrow \bar{\beta}\cdot\lambda = \sum \frac{\bar{\alpha}_i}{|\beta-\alpha_i|^2} \Rightarrow \beta = \frac{1}{\lambda} \sum \frac{\alpha_i}{|\beta-\alpha_i|^2}$$

$$\text{Let } \frac{1}{|\beta-\alpha_i|^2} = \lambda_i$$

$$\Rightarrow \boxed{\beta = \sum \left(\frac{\lambda_i}{\lambda}\right) \alpha_i} \qquad \sum \frac{\lambda_i}{\lambda} = 1 \ ; \ \frac{\lambda_i}{\lambda} \geq 0$$

convex hull.

Hence β is the convex combination of $\lambda_1, \lambda_2, \ldots, \lambda_n$.
Hence we can say β lies in the convex hull of the set of zeros of $p_n(z)$.

Hence proved.

Proof of (2):- Let $p(z)$ be a monic polynomial of degree n with roots $\alpha_1, \alpha_2, \ldots, \alpha_n$.

$$p(z) = (z-\alpha_1)(z-\alpha_2)\cdots(z-\alpha_n)$$

Taking log on both sides and differentiating

$$\frac{p'(z)}{p(z)} = \sum \frac{1}{(z-\alpha_i)}$$

$$= \sum \frac{\bar{z}-\bar{\alpha_i}}{|z-\alpha_i|^2}$$

$\because z = x+iy$
$\alpha_i = a_i + ib_i$
and $z.\bar{z} = |z|^2$

$$= \sum \frac{(x-iy)-(a_i-ib_i)}{|z-\alpha_i|^2}$$

$$= \sum \frac{(x-a_i)-i(y-b_i)}{|z-\alpha_i|^2}$$

Let $p'(z)=0$ for $z = \alpha+i\beta$. Then

$$\frac{p'(z)}{p(z)} = \sum \frac{(\alpha-a_i)-i(\beta-b_i)}{|\beta|^2}$$

then α and a_i have same sign & β and b_i have also same sign.

Hence proved

Result :- Any non-constant, entire function can not have a set of zeros, which has limit point in the finite part of the complex plane.

Problem :- $f: \mathbb{C} \to \mathbb{C}$ such that $f(z)$ is non-constant and entire then which one is possible :-

(i) $f(z)$ is bounded

(ii) 0 is limit point

(iii) $f(n) = n \;\forall\; n \in \mathbb{Z}$

Solution :- (iii) is possible

$\because$ If $f(z) = z$

then $f(n) = n \;\forall\; n \in \mathbb{Z}$.

Result :- $f(z) = e^z$ is many-valued i.e. not one-one then we cannot find any nbd in which it is univalent.

Note (i) If $f(z)$ is entire and $\lim\limits_{z \to \infty} f(z) \to \infty$ then $f(z)$ is non-constant.

(ii) If $f(z)$ is analytic and $\lim\limits_{z \to \infty} f(z) \to 1$ then $f(z)$ is constant.

$\because |f(z)| \le M$ for $|z| > R$ i.e. bdd

i.e. If $f(z)$ is entire and have finite limit as $z \to \infty$ then $f(z)$ is constant.

(iii) If $f(z)$ is entire and $f(z)$ is continuous on $|z| \le R$ ($f(z)$ is bdd on $|z| \le R$)

$f(z)$ is constant.

V. Imp. $\to$ Second name of pole is non-essential singularity.

→ $f(z) = \begin{cases} \frac{1}{z} & ; z \neq 0 \\ 0 & ; z = \infty \\ \infty & ; z = 0 \end{cases}$ (not diff.)

$\lim_{z \to a} f(z)$ exists infinitely if for any region $R \in \mathbb{R}$ $\exists$ $\delta > 0$

s.t. $|f(z)| > R$ if $|z-a| < \delta$

then $\lim_{z \to a} f(z) = \infty$

→ $f(z)$ is differentiable when limit exists finitely.

$\boxed{\text{Note}}$: (i) Everywhere means $\mathbb{C}$ complex plane not extended complex plane.

(ii) Differentiation is a point base property.

$\boxed{\text{Regular point}}$:- $z_0 \in \mathbb{C}$ is said to be regular point of $f(z)$ if $\exists$ a δ-nbd of z_0 s.t. $f(z)$ is differentiable at every point of that nbd.

→ nbd of z_0 will be a disc.

$\boxed{\text{e.g.}}$:- $f(z) = |z|^2$ is differentiable only at $z=0$.

$f(z) = x^2+y^2$

$\therefore u = x^2+y^2, v=0$

So $u_x = v_y$ and $u_y = -v_x$ holds (by C-R eqn)

only at $z=0$.

So $f(z) = |z|^2$ is diff. only at $z=0$

But $z=0$ is not a regular point of $f(z)$.

$\boxed{\text{Analyticity}}$:- If $f(z)$ is analytic in domain D then every point of D is a regular point of $f(z)$.

NO boundary.

ϕ

Singularity :- If $f(z)$ is not differentiable at z_0 but z_0 is limit point of regular points of $f(z)$ then z_0 is said to be singularity of $f(z)$.

e.g. :- $f(z) = |z|^2$ has no singularity because $f(z)$ has no regular point. $f(z)$ is differentiable at $z=0$ only.

→ If z_0 is a limit point of regular points and $f'(z_0)$ exists then $\exists$ a nbd of z_0 in which $f(z)$ is analytic. z_0 is regular point of $f(z)$.

e.g. :- $f(z) = \sin x$

$u = \sin x \Rightarrow u_x = \cos x$
$v = 0 \quad\quad v_y = 0$ } C-R equ^n does not satisfy

$\Rightarrow$ $f(z)$ is not differentiable at $z=0$.

e.g. :- If $g(z) = u+iv$ is entire

$\Rightarrow f(z) = \arg g(z) = \tan^{-1}\left(\frac{v}{u}\right)$

No regular point

$\Rightarrow$ No singularity [$\because$ singularity is a l.p. of regular pts

→ Function is not differentiable at singular point.

Note :- If a function has no regular points then it has no singularity.

→ If $f(z) = u+iv$ is analytic on D

$\Rightarrow \boxed{\frac{\partial f}{\partial \bar{z}} = 0} \quad \forall z \in D$

↳ complex form of C-R equations.

e.g.: $f(z) = \sin\bar{z}$

$$\frac{\partial f}{\partial \bar{z}} = \cos\bar{z}$$

$$\because \frac{\partial f}{\partial \bar{z}} \neq 0$$

It has no regular point.

$\Rightarrow$ It has no singularity then $f(z)$ is nowhere analytic.

e.g.:- $f(z) = z \cdot \bar{z}$

$$\frac{\partial f}{\partial \bar{z}} = z$$

$$\because \frac{\partial f}{\partial \bar{z}} \neq 0$$

It has no regular point

$\Rightarrow$ It has no singularity then $f(z)$ is nowhere analytic.

e.g.:- Find the singularity of

$$f(z) = \frac{\sin 2z - 2\sin z}{(\sin z - z)\sin z}$$

Solution:- $f(z) = \dfrac{\sin 2z - 2\sin z}{(\sin z - z)\sin z}$

$$= \frac{2\sin z\cos z - 2\sin z}{(\sin z - z)\sin z}$$

$$= \frac{2\cancel{\sin z}(\cos z - 1)}{(\sin z - z)\cancel{\sin z}}$$

$$f(z) = \frac{2(\cos z - 1)}{\sin z - z}$$

$$\lim_{z\to 0} f(z) = \lim_{z\to 0} \frac{2(\cos z - 1)}{(\sin z - z)} \quad \left[\frac{0}{0} \text{ form}\right]$$

$$= \lim_{z \to 0} \frac{-2\sin z}{\cos z - 1} \quad \text{[By L-Hospital rule]} \quad \left[\frac{0}{0}\right]$$

$$= \lim_{z \to 0} \frac{2\cos z}{\sin z}$$

$$= \frac{2 \lim_{z \to 0} \cos z}{\lim_{z \to 0} \frac{\sin z}{z}} . z = 1 . 2 . 1 = 2 \neq 0$$

$\Rightarrow z = 0$ is simple pole.

$\boxed{C_{15}}$:- If $f(z)$ is analytic in D s.t. max $|f(z)|$ attains somewhere in D then $f(z)$ is constant.

UNIT-3 :-

Meromorphic function :- A function is said to be meromorphic function if it has no essential singularity in the complex plane $\mathbb{C}$

i.e. infinite is the only possible point where it can have extended essential singularity.

OR. An analytic function whose only singularities in the finite complex plane are poles, is called meromorphic function.

OR. A function is said to be meromorphic if it is free from essential singularity in the finite part of the complex plane.

i.e. $z=\infty$ is only possible essential singularity.

e.g :- V.imp.
(i) Entire functions are meromorphic

(ii) $f(z) = \frac{1}{z}$

It has only one non-essential singularity (i.e. pole) which is pole of order 1 i.e. simple pole.

Rational function :- A complex valued function is said to be rational function if poles are only singularities in the extended complex plane [i.e. $\mathbb{C} \cup \{\infty\}$].

By redefining the removable singularity has been removed and rational function always in the form $f(z) = \frac{p(z)}{q(z)}$

where $p(z)$ and $q(z)$ are polynomials

i.e. no essential singularity in the extended complex plane.

Note :- (i) Every rational function is meromorphic but every meromorphic function need not be rational function.

i.e. $\boxed{\text{Rational function} \Rightarrow \text{Meromorphic function}}$

(ii) If $f(z)$ is meromorphic, then $z=\infty$ is the only possible point for non-isolated essential singularity.

(iii) Rational function can't have non-isolated singularity in the extended complex plane

i.e. a rational function has only finite no. of singula

Imp.
(iv) All the entire functions are meromorphic but conve is not true.

i.e. $\boxed{\text{Entire function} \Rightarrow \text{Meromorphic function}}$

e.g. :- $\underbrace{\sin z, \cos z, e^z}_{\text{Entire}}$ are meromorphic functions

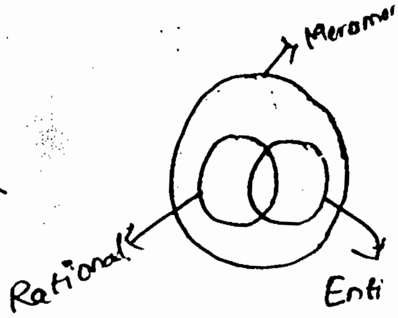
But $\sin\frac{1}{z}, \cos\frac{1}{z}$ are not entire functions.

e.g. :- (i) $e^z \longrightarrow$ E, M, Not R

(ii) $\dfrac{e^z}{\sin z} \longrightarrow$ Not E, M, Not R

(iii) $\dfrac{\sin z}{e^z} \longrightarrow$ E, M, Not R

(iv) $\sin z + \dfrac{1}{z-a} \longrightarrow$ Not E, M, Not R



$\boxed{\begin{array}{l} z=\infty \to \text{essential} \\ z=a \to \text{pole} \end{array}}$

(v) If an entire function is rational then it is a polynomial.

(vi) A function which is not rational is called transcedenta function.

$\rightarrow$ $f(z) = z, z^2$ i.e. polynomials (only numerator) Meromophic + E + R

Result: An entire function $f(z)$ is transceedental iff $f(1/z)$ has essential singularity at $z=0$.

Result:- Any meromorphic function can have only finite number of zeros or poles in any closed and bdd region.

14 **Result**:- If an entire function has no singularity in the extended complex plane then it is constant.

Proof:- $f(z) = \sum_{n=0}^{\infty} a_n z^n = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n + \cdots$

$$f\left(\frac{1}{z}\right) = a_0 + \frac{a_1}{z} + \frac{a_2}{z^2} + \cdots$$

$\Rightarrow f(1/z)$ has singularity at $z=0$

$\Rightarrow f(z)$ " " at $z=\infty$

But $f(z)$ is analytic at $z=\infty$

$\Rightarrow a_1 = a_2 = \cdots a_n = 0$

Then $\boxed{f(z) = a_0}$

Hence $f(z)$ is constant. Hence proved

→ An entire function is said to be transceedental iff it has only essential singularity at $z=\infty$.

→ $\sin z$ is entire, It has only possible singularity at $z=\infty$ so $\sin z$ is meromorphic. But it is not rational because that singularity at ∞ is other than pole ($\sin z$ has non-isolated singularity at ∞).

→ $\frac{\sin z}{\cos z}$ → It has singularity at $z=\infty$.

Meromorphic but not rational.

Expansion of Analytic function :- We can expand $f(z)$ in the nbd of $z=a$ in terms of value of $f(z)$ in the nbd of a.

We have two types of expansion:-

(I) **Taylor's Expansion** :- Let $z=a$ be the point of expansion then Taylor's expansion is obtained if $f(z)$ is regular at 'a'.

i.e. 'a' is the regular point of $f(z)$.

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n \quad \text{———} \quad (i)$$

$$\text{Where } a_n = \frac{f^n(a)}{n!} = \frac{1}{\cancel{n!}} \frac{\cancel{n!}}{2\pi i} \int_C \frac{f(z)}{(z-a)^{n+1}} dz$$

$$\text{i.e. } a_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-a)^{n+1}} dz \qquad [C: |z-a| = r]$$

Then the series on R.H.S. of equ^n (i) is uniformly convergent in that nbd of 'a' and converges to $f(z)$

Note: $$\int_C (z-a)^n dz = \begin{cases} 0 & ; \text{if } n \neq -1 \\ 2\pi i & ; \text{if } n = -1 \end{cases} \quad ; C: |z-a| = r$$

(II) **Laurent's Expansion** :- Let $z=a$ be an isolated singularity of $f(z)$ then in the nbd of this point Laurent expansion is obtained as

$$f(z) = \underbrace{\sum_{n=0}^{\infty} a_n (z-a)^n}_{\text{Regular part (A)}} + \underbrace{\sum_{n=1}^{\infty} b_n (z-a)^{-n}}_{\text{Principal part (B)}} \quad \text{———} \quad (ii)$$

The series on R.H.S. converges uniformly in some deleted nbd of 'a'

i.e. $0 < |z-a| < r$ to $f(z)$

and $a_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-a)^{n+1}} dz$

$b_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-a)^{-n+1}} dz$

$C: |z-a| < r_1 < r$

and $f(z)$ does not include any singularity in this deleted nbd of 'a'.

Ist part of equ^n (ii) is called regular part and IInd part is called principal part.

$\boxed{\text{Note}}$:- $f(z) = \sum_{n=0}^{\infty} \left(\frac{z}{3}\right)^n + \sum_{n=1}^{\infty} \frac{1}{z^n}$

$\left|\frac{z}{3}\right| < 1$ & $\left|\frac{1}{z}\right| < 1$

$\Rightarrow |z| < 3$ & $|z| > 1$

$\Rightarrow 1 < |z| < 3.$

$f(z) = \frac{1}{1-\frac{z}{3}} + \frac{1}{z-1}$

$= \frac{3}{3-z} + \frac{1}{z-1} = \frac{3(z-1)+(3-z)}{(3-z)(z-1)}$

$\Rightarrow f(z) = \frac{2z}{(3-z)(z-1)}$

$z=1$ & $z=3$ are singularities of $f(z)$.

Types of Singularity with the help of Laurent's expansion

If $\underbrace{\sum_{n=0}^{\infty} a_n(z-a)^n}_{\text{Regular part}} + \underbrace{\sum_{n=1}^{\infty} b_n(z-a)^{-n}}_{\text{principal part}}$ converges uniformly in $0<|z-a|<r$ to $f(z)$ then

(i) $z=a$ is either removable singularity or regular point of $f(z)$ if $b_n=0 \; \forall \; n \in \mathbb{N}$

i.e. there is no principal part.

(ii) If $\exists \; m \in \mathbb{N}$ s.t. $b_n=0 \; \forall \; n>m$ & $b_m \neq 0$ then

i.e. principal part contains finite number of terms

$z=a$ is pole of order m.

OR There are finite number of terms in the principal part i.e. at least one of b_n is non-zero and highest power of $\left(\frac{1}{z-a}\right)$ is order of pole.

(iii) If principal part contains infinite number of terms then $z=a$ is essential singularity

i.e. $b_n \neq 0$ for infinite values of n.

Note:- If $f(z)$ has isolated singularity at $z=a$, then one should take help of Taylor's expansion of $f(z)$ to classify the singularity of $f(z)$.

e.g.:- ① $f(z) = e^{1/z} = 1+\frac{1}{z}+\cdots$

$z=0$ is an isolated essential singularity.

(ii) $f(z) = \frac{\sin z}{z^2}$

(iii) $f(z) = \frac{\cos z}{z^2}$

(iv) $\int \frac{\sin z}{z^{2009}} dz = 0$

(v) $\int_{|z|=1} \frac{\cos z}{z^{2008}} = 0$

(vi) $\int_{|z|=1} \frac{\cos z}{z^{2009}} dz = 2\pi i \frac{z^{2008}}{\lfloor 2008} \times \frac{1}{z^{2009}}$

$= \frac{2\pi i}{\lfloor 2008}$

Radius of convergence for Taylor's and Laurent's series :-

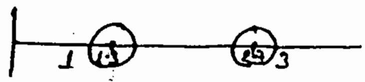
Radius of convergence is distance of nearest singularity from point of expansion.

e.g. :- $f(z) = \dfrac{1}{(z-1)(z-3)}$

$z = 1.3$. then radius of convergence (roc) $= 0.3$

$z = 2.7$ " " " " " $= 0.3$

$z = 2$ " " " " " $= 1$



Extension of Liouville's Theorem :- If entire function f satisfies the condition $|f(z)| \le M.|z|^{\alpha}$

where $M > 0$, $\alpha \geq 0$

i.e $|f(z)|$ increases more slowly than some power of $|z|$ as $z \to \infty$ $\forall$ $z \in \mathbb{C}$, then $f(z)$ is a polynomial of degree n, where 'n' is the ~~least~~ largest integer less than or equal to α.

Proof :- If $\alpha = 0$ gives the Liouville's theorem.

Now $f(z)$ is entire, we can have Taylor's expansion

$$f(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n + \cdots$$

$$a_k = \frac{f^k(0)}{k!}$$

Now, $f^k(0) = \dfrac{k!}{2\pi i} \displaystyle\int_C \frac{f(z)}{z^{k+1}}\, dz$

$\Rightarrow \left| f^{k-1}(0) \right| = \left| \dfrac{(k-1)!}{2\pi} \displaystyle\int_C \frac{f(z)}{z^k}\, dz \right|$

$$\leq \frac{(k-1)!}{2\pi} \int_C \frac{|f(z)|}{|z|^k} |dz|$$

$$\leq \frac{(k-1)!}{2\pi} \int_C \frac{M.|z|^\alpha}{|z|^k} dz$$

i.e. $$|f^{k-1}(0)| \leq \frac{(k-1)!}{2\pi} M \int \frac{1}{|z|^{k-\alpha}} dz$$

$$\leq \frac{(k-1)!}{2\pi} \frac{M}{R^{k-\alpha}} \int_C |dz| \quad ; \; |z| = R$$

$$\leq \frac{(k-1)!}{\cancel{2\pi}} \times \frac{M}{R^{k-\alpha}} \cdot \cancel{2\pi} R$$

$$\leq (k-1)! \; M R^{\alpha-(k-1)}$$

If $k-1 > \alpha$ then $|f^{k-1}(0)| \to 0$ as $R \to \infty$

Hence a_r the co-efficient in Taylor's series is zero

where $r \geq k-1 > \alpha$

that means $f(z)$ is polynomial of degree n where n is the largest integer less than or equal to α.

Hence proved

Application of Taylor and Laurent's series :-

(I). Let $z=a$ be an isolated singularity of $f(z)$ then by Laurent's series

$$f(z)=\sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} b_n (z-a)^{-n} \quad ; \quad 0<|z-a|<r$$

$$\frac{1}{2\pi i}\int_C f(z)\,dz = \frac{1}{2\pi i}\int_C \sum_{n=0}^{\infty} a_n (z-a)^n dz + \frac{1}{2\pi i}\int_C \sum_{n=1}^{\infty} b_n (z-a)^{-n} dz$$

$$= \sum_{n=0}^{\infty} a_n \cdot \frac{1}{2\pi i}\int_C (z-a)^n dz + \sum_{n=1}^{\infty} b_n \cdot \frac{1}{2\pi i}\int_C (z-a)^{-n} dz$$

Note :- $\boxed{\int_C (z-\alpha)^n dz = \begin{cases} 2\pi i & ; n=-1 \\ 0 & ; \text{otherwise} \end{cases}}$

$$= b_1$$

i.e. $\frac{1}{2\pi i}\int_C f(z)\,dz = b_1$

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Note :- The residue is defined only at isolated singularity.

$$\boxed{\int_C f(z)\,dz = 2\pi i b_1} \qquad C: |z-a|=r$$

(II) **Cauchy's Residue Theorem** :- If $f(z)$ is analytic on C where C is closed contour and with in C, there are isolated singularities then

$$\boxed{\int_C f(z)\,dz = 2\pi i \times (\text{sum of residues at all the singularities within } C)}$$

(III) Residue at infinity :- If $f(z)$ has isolated singularity at $z=\infty$ and in the finite part of complex plane there are finite number of singularities then residue at ∞ is defined as

$$\boxed{\frac{1}{2\pi i}\int_C f(z)\,dz = Res(z=\infty)}$$

When C travells clockwise direction and conteins all the singularities of $f(z)$.

$$\boxed{Res(z=\infty) = -\frac{1}{2\pi i}\int_C f(z)\,dz} \rightarrow \text{anticlock direction}$$

$\int_C f(z)\,dz = -2\pi i \times$ sum of residues at all the singularities in C. Now C is anticlockwise direction.

Note (1) :- In the finite part of the complex plane at any regular point residue always zero but at ∞ even it is regular point we may get non-zero residue.

e.g :- $f(z) = \frac{1}{z}$ is analytic at ∞

But residue is at $\infty = -\frac{1}{2\pi i}\int_C \frac{dz}{z} = \frac{-1}{2\pi i} \times 2\pi i = -1.$

Note (2) :- If $f(z)$ is regular in D and has zeros of order m at $a \in D$, then $\exists\ g(z)$ analytic in nbd of 'a'.

where $g(z) \neq 0$ for any $g(z) \neq 0$ $z \in |z-a| < \delta$ s.t. $f(z) = (z-a)^m g(z)$

$g(a) \neq 0$

NOW, $f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n$

$$= a_0 + a_1(z-a) + a_2(z-a)^2 + \dots + a_{m-1}(z-a)^{m-1} + a_m(z-a)^m + \dots$$

$$a_i = \frac{f^i(a)}{i!}$$

$f(a) = 0 \Rightarrow f^i(a) = 0$ for $i = 0$ to $m-1$

$$f(z) = a_m(z-a)^m + a_{m+1}(z-a)^{m+1} + \dots$$

$$= (z-a)^m [a_m + a_{m+1}(z-a) + \dots]$$

$$= (z-a)^m [b_0 + b_1(z-a) + b_2(z-a)^2 + \dots]$$

where $b_i = a_{m+i}$; $i = 0, 1, 2, \dots$

$$f(z) = (z-a)^m \left[\sum_{i=0}^{\infty} b_i (z-a)^i\right]$$

$g(z) = \sum b_i (z-a)^i \Rightarrow f(z) = (z-a)^m g(z)$.

$g(a) = b_0 = a_m \neq 0$.

$g(z)$ is analytic ; $g(a) \neq 0$.

$\boxed{\text{Note (3)}}$:- To show zeros are isolated.

$\boxed{\text{Proof}}$:- Let $z = a$ is zeros of order m of $f(z)$ then by previous theorem $\exists$ $g(z)$ and a nbd $N_\delta : |z-a| < \delta$ s.t. $f(z) = (z-a)^m g(z)$ where $g(a) \neq 0$ and $g(z)$ is analytic in N_δ.

Now let in every nbd of 'a' there is a zeros of $g(z)$.

Hence for any $n \in \mathbb{N}$ $\exists$ $z_n \in N_\delta$; $|z-a| < 1/n$

s.t. $g(z_n) = 0$

$\langle z_n \rangle \longrightarrow a$

and $\langle g(z_n) \rangle \longrightarrow 0 \neq g(a)$

as $g(a) = a_m \neq 0$.

$g(z)$ is not analytic at $z = a$ <u>Contradiction</u>.

Hence $\exists\ \delta$ s.t. $g(z) \neq 0$

for any $z \in D$: $|z-a| < \delta$

then $f(z) = (z-a)^m g(z)$ has only one zero $z=a$ in $|z-a| < \delta$

Hence zeros are isolated.

Hence proved.

Result :- If $f(z)$ has pole of order m at $z=b$ then $f(z)$ can be written as

$$f(z) = \frac{\phi(z)}{(z-b)^m} \quad \text{where } \phi(b) \neq 0$$

and $\exists\ \delta > 0$ s.t. $\phi(z)$ is analytic in $|z-b| < \delta$.

Converse of the theorem is also true.

Proof :- As $z=b$ is pole of order m.

By Laurent's series.

$$f(z) = \sum a_n (z-b)^n + \frac{b_1}{z-b} + \frac{b_2}{(z-b)^2} + \cdots\cdots$$

$$\Rightarrow (z-b)^m f(z) = \sum_{n=0}^{\infty} a_n (z-b)^{m+n} + b_1 (z-b)^{m-1} + b_2 (z-b)^{m-2} + \cdots\cdots$$

$$\cdots + b_{m-1}(z-b) + b_m$$

$$= \sum_{i=0}^{\infty} c_i (z-b)^i \quad \text{where } c_i = \begin{cases} b_{m-i} & i = 0 \text{ to } m-1 \\ a_{i-m} & i \geq m \end{cases}$$

$$= \phi(z) \quad \text{and } \phi(b) = b_m \neq 0.$$

then $\exists\ N_\delta$ s.t. $\phi(z)$ is analytic in N_δ and $\phi(b) \neq 0$

$$\Rightarrow (z-b)^m f(z) = \phi(z)$$

$$\Rightarrow \boxed{f(z) = \frac{\phi(z)}{(z-b)^m}}$$

Hence proved

Converse :- If $f(z) = \dfrac{\phi(z)}{(z-b)^m}$ where $\phi(b) \neq 0$

where b is regular point of $\phi(z)$

then $z=b$ is a pole of $f(z)$ of order m.

Proof :- $\phi(z)$ is analytic in a N_δ of $z=b$

Then by Taylor's series

$$\phi(z) = \sum_{n=0}^{\infty} C_n (z-b)^n$$

then $$f(z) = \frac{\phi(z)}{(z-b)^m} = \frac{1}{(z-b)^m} \sum_{n=0}^{\infty} C_n (z-b)^n$$

$$= \underbrace{\frac{C_0}{(z-b)^m} + \frac{C_1}{(z-b)^{m-1}} + \cdots + \frac{C_{m-1}}{(z-b)}}_{\text{principal part}} + C_m + C_{m+1}(z-b)$$

$$+ C_{m+2}(z-b)^2 + \cdots$$

Regular part

$z=b$ is pole as there are only finite number of terms in the principal part as $C_0 = b_m \neq 0$

the order of pole is m.

as $\text{Res}(z=b)$ = co-efficient of $\frac{1}{(z-b)}$

$$= C_{m-1} = b_1$$

$$b_1 = C_{m-1} = \frac{\phi^{m-1}(b)}{(m-1)!} = \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} \phi(z) \text{ at } z=b$$

$$\boxed{b_1 = \frac{1}{(m-1)!} \lim_{z \to b} \frac{d^{m-1}}{dz^{m-1}} (z-b)^m f(z)} \rightarrow \text{V.V. Imp}$$

e.g. :- $f(z) = \dfrac{\cos z}{z^3}$ Here $m=2$

$$\therefore \text{Res}(z=0) = \frac{1}{2!} \lim_{z \to 0} \frac{d^2}{dz^2} \left(z^3 \cdot \frac{\cos z}{z^3} \right) = \frac{1}{2} (-\cos z) \Big|_{z=0} = \frac{-1}{2}$$

The Calculus Of Residues :-

Definition :- **Residue at a pole** :- Suppose a single valued function $f(z)$ has a pole of order m at $z=a$, then by definition of pole, the principal part of Laurent's expansion of $f(z)$ contains only m terms so that

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} b_n (z-a)^{-n} \quad \text{——— (1)}$$

Where $a_n = \frac{1}{2\pi i} \int_C \frac{f(z)\,dz}{(z-a)^{n+1}}$, $b_n = \frac{1}{2\pi i} \int_C \frac{f(z)\,dz}{(z-a)^{-n+1}}$

C being a circle $|z-a| = r$

Evidently $b_1 = \frac{1}{2\pi i} \int_C f(z)\,dz$ ——— (2)

The co-efficient b_1 is called the residue of $f(z)$ at the pole $z=a$ and is denoted by the symbol $Res(z=a)$.

Thus $\boxed{Res(z=a) = b_1}$.

Evidently the value of b_1, given by (2), does not depend upon the order of the pole and hence it represents a general definition of the residue at a pole.

Consider the case in which $z=a$ is a simple pole i.e. $z=a$ is a pole of order 1.

Then (1) becomes

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \frac{b_1}{(z-a)}$$

$$\Rightarrow \lim_{z \to a} (z-a) f(z) = b_1$$

Using (2), we get

$$\boxed{Res(z=a) = \lim_{z \to a} (z-a) f(z) = b_1 = \frac{1}{2\pi i} \int_C f(z)\,dz}$$

Here the circle C may be replaced by any closed contour containing within it no other singularity except $z=a$.

Definition: **Residue at infinity** :- If $f(z)$ has an isolated singularity at $z=\infty$ or is analytic there, then the residue at $z=\infty$ is defined as

$$\boxed{Res\,(z=\infty) = -\frac{1}{2\pi i}\int_C f(z)\,dz}$$

Where C is any closed contour which encloses all the finite singularities of $f(z)$. The integral is taken in +ve direction (anticlockwise direction).

Remark :- (i) The function may be regular at infinity, yet has a residue there.

Consider the function $f(z) = \frac{b}{z-a}$

For this function

$$Res\,(z=\infty) = -\frac{1}{2\pi i}\int_C f(z)\,dz = \frac{-1}{2\pi i}\int_C \frac{b}{z-a}\,dz$$

$$= \frac{-b}{2\pi i}\int_0^{2\pi} \frac{r i e^{i\theta}\,d\theta}{re^{i\theta}} = -\frac{b}{2\pi}\int_0^{2\pi} d\theta = -b$$

Where C is the circle $|z-a| = r$

$\therefore$ $\boxed{Res\,(z=\infty) = -b}$

Also $z=a$ is a simple pole of $f(z)$ and its residue there is

$$\frac{1}{2\pi i}\int_C f(z)\,dz = b$$

Therefore $\boxed{Res\,(z=a) = b = -Res\,(z=\infty)}$

(ii) If a function is analytic at a point $z=a$, then its residue at $z=a$ will be zero, but not so at infinity.

(iii) If $f(z) = \dfrac{(z^2+2z+7)\,e^z}{(z-5)^2(z-2)^7(z-9)^3}$, then $f(z)$ has poles at $z = 5, 2, 9$ of orders 2, 7, 3 respectively.

Theorem:- **Cauchy's Residue Theorem**:- If $f(z)$ is analytic within and on a closed contour C, except at a finite number of poles $z_1, z_2, z_3 \ldots z_n$ within C, then

$$\boxed{\int_C f(z)\,dz = 2\pi i \sum_{r=1}^{n} \text{Res}(z=z_r)}$$

Where R.H.S. denotes sum of residues of $f(z)$ at its poles lying within C.

V.V.I.

Result:- If a function $f(z)$ is analytic except at finite number of singularities (including that at infinity), then the sum of residues of these singularities is zero.

Problem:- Evaluate the residues of $\dfrac{z^2}{(z-1)(z-2)(z-3)}$ at 1, 2, 3, and infinity and show that their sum is zero.

Solution:- Let $f(z) = \dfrac{z^2}{(z-1)(z-2)(z-3)}$

$$\text{Res}(z=1) = \lim_{z\to 1}(z-1)f(z) = \lim_{z\to 1}\frac{z^2}{(z-2)(z-3)} = \frac{1}{(1-2)(1-3)} = \frac{1}{2}.$$

$$\text{Res}(z=2) = \lim_{z\to 2}(z-2)f(z) = \lim_{z\to 2}\frac{z^2}{(z-1)(z-3)} = \frac{4}{(2-1)(2-3)} = \frac{4}{1\times(-1)} = -4$$

$$\text{Res}(z=3) = \lim_{z\to 3}(z-3)f(z) = \lim_{z\to 3}\frac{z^2}{(z-1)(z-2)} = \frac{9}{(3-1)(3-2)} = \frac{9}{2}$$

$$\text{Res}(z=\infty) = \lim_{z\to\infty} -z\,f(z) = \lim_{z\to\infty}\frac{-z^3}{(z-1)(z-2)(z-3)} = -1$$

$$\text{Sum of residues} = \frac{1}{2} - 4 + \frac{9}{2} - 1 = \underline{\underline{0}}$$

Similar Problem:- The residue of $\dfrac{z^3}{(z-1)(z-2)(z-3)}$ at $z = 1, 2, 3$ are respectively

• (A) $1/2, -8, 27/2$ (B) $1, -8, 27/2$ (C) $1/2, 0, 27/2$ (D) none of these

Similar problem :- The number of poles of $f(z) = \frac{1}{z(z^2+3)(z^2+2)^3}$ inside the circle $|z|=1$ are:

(A) 1 (B) 9 (C) 5 (D) 2.

Similar problem :- Evaluate the residue of $\frac{z^3}{(z-1)(z-2)(z-3)}$ at $z=\infty$.

Solution :- We expand the function in the nbd of $z=\infty$ as follows:

$$f(z) = \frac{z^3}{(z-1)(z-2)(z-3)}$$

$$= \left(1-\frac{1}{z}\right)^{-1}\left(1-\frac{2}{z}\right)^{-1}\left(1-\frac{3}{z}\right)^{-1}$$

$$= \left(1+\frac{1}{z}+\cdots\right)\left(1+\frac{2}{z}+\cdots\right)\left(1+\frac{3}{z}+\cdots\right)$$

$$= 1+\frac{6}{z} + \text{higher powers of } \frac{1}{z}$$

Hence the residue at infinity $= -6$ = negative co-eff. of $\frac{1}{z}$.

Similar problem :- Evaluate the residues of $f(z)$ where $f(z) = \frac{e^z}{z^2(z^2+9)}$ at $z=0, -3i, +3i$.

Solution :- Here $f(z) = \frac{e^z}{z^2(z^2+9)}$

Its poles are $z=0, -3i, +3i$

Since $z=0$ is the pole of 2nd order, so

$$\operatorname*{Res}_{z=0} f(z) = \frac{1}{1!}\frac{d}{dz}\left(\frac{e^z}{z^2+9}\right) \text{ at } z=0$$

$$= \frac{e^z\cdot(z^2+9) - e^z\cdot 2z}{(z^2+9)^2} \text{ at } z=0$$

$$= \frac{1}{9}$$

$z=-3i$ is a simple pole.

$$\operatorname*{Res}_{z=-3i} f(z) = \lim_{z\to-3i}\frac{(z+3i)e^z}{z^2(z^2+9)} = \lim_{z\to-3i}\frac{e^z}{z^2(z-3i)}$$

$$= \frac{e^{-3i}}{(-3i)^2(-6i)} = \frac{-ie^{-3i}}{54}.$$

Similarly, $\underset{z=3i}{\text{Res}}\, f(z) = \dfrac{ie^{3i}}{54}.$

Problem:- Evaluate by the method of calculus of residues:-

$$\int_C \frac{dz}{(z-1)(z+1)}$$

Where C is circle $|z| = 3$.

Solution:- Let $f(z) = \dfrac{1}{(z-1)(z+1)}$.

Poles of $f(z)$ are given by $(z-1)(z+1) = 0$ or, $z = 1, -1$. These are simple poles and lie within C.

$$\text{Res}(z=1) = \lim_{z\to 1}(z-1)f(z) = \lim_{z\to 1} \frac{\cancel{(z-1)}}{\cancel{(z-1)}(z+1)}$$

$$= \lim_{z\to 1} \frac{1}{z+1} = \frac{1}{2}$$

$$\text{Res}(z=-1) = \lim_{z\to -1}(z+1)f(z) = \lim_{z\to -1} \frac{1}{z-1} = -\frac{1}{2}$$

$$\text{Res}(z=1) + \text{Res}(z=-1) = \frac{1}{2} - \frac{1}{2} = 0$$

$$\int_C f(z)\,dz = 2\pi i \times (\text{sum of residues}) = 2\pi i \times 0 = \underline{\underline{0}} \text{ Ans.}$$

Problem:- Using residue theorem, evaluate $\displaystyle\int_C \frac{e^z\,dz}{z(z-1)^2}$

Where C is circle $|z| = 2$.

Solution:- Let $I = \displaystyle\int_C \frac{e^z\,dz}{z(z-1)^2}$, where C is circle $\underline{\underline{|z| = 2}}$.

Here centre is $z = 0$ and radius $= 2$

$\therefore$ $z = 0, 1$ are poles lying within C.

$z=0$ is a simple pole.

$$Res(z=0) = \lim_{z\to 0} (z-0) f(z) = \lim_{z\to 0} z\left[\frac{e^z}{z(z-1)^2}\right]$$

$$= \lim_{z\to 0} \frac{e^z}{(z-1)^2} = \frac{e^0}{(0-1)^2} = 1.$$

$z=1$ is a pole of order 2.

Take $f(z) = \dfrac{\phi(z)}{(z-1)^2}$; where $\phi(z) = \dfrac{e^z}{z}$

$$Res(z=1) = \frac{\phi'(1)}{1!} \quad ; \quad \phi'(z) = \frac{e^z . z - 1 . e^z}{z^2}$$

$$\phi'(1) = \frac{e^1 . 1 - 1 . e^1}{1^2} = 0$$

$\therefore$ $Res(z=1) = 0.$

By Cauchy's residue theorem,

$$I = 2\pi i\,(\text{sum of residues within } C)$$

$$= 2\pi i\,[Res(z=0) + Res(z=1)]$$

$$= 2\pi i\,[1+0]$$

$$= \underline{\underline{2\pi i}} \text{ Ans}$$

Result :- If an analytic function $f(z)$ has a pole at $z=\infty$, then the residue of $f(z)$ at infinity is the negative of the coefficient of $\frac{1}{z}$ in the expansion of $f(z)$ for the values of z in the nbd of $z=\infty$.

Result :.
$$\boxed{\lim_{z\to\infty} -z f(z) = Res(z=\infty) \quad \text{provided } f(z) \text{ is analytic at } z=\infty.}$$

Computation of Residue at a finite pole :-

1). Residue of $f(z)$ at a simple pole $z=a$.

(i) $Res(z=a) = \lim_{z \to a} (z-a) f(z)$

(ii) Let $f(z) = \dfrac{\phi(z)}{\psi(z)}$ have a simple pole at $z=a$.

2). Residue at a pole of order m.

Result :- If $f(z)$ has a pole of order m at $z=a$, then the residue at 'a' is the limit of

$$\frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} \left[(z-a)^m f(z) \right] \text{ as } z \to a.$$

OR, The residue of $\dfrac{\phi(z)}{(z-a)^m}$ at $z=a$ is $\dfrac{\phi^{(m-1)}}{(m-1)!}$

i.e. $$\boxed{Res(z=a) = \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} \left[(z-a)^m f(z) \right] \text{ as } z \to a.}$$

Result :- **Liouville's Theorem** :- If function is analytic at every point and finite at infinity, then it must be constant.

3). Residue at a pole $z=a$ of any order.

We have seen that the residue of $f(z)$ at $z=a$ is the co-efficie of $\frac{1}{z-a}$ in Laurent's expansion of $f(z)$ and therefore co-efficient of $\frac{1}{t}$ in the expansion of $f(a+t)$ as a power series.

V.V. Imp

[S.K. RATHORE]

Extra (VIII)

Working Rule (for computation the residue)

1). $\text{Res}(z=a) = \lim_{z \to a} (z-a) f(z)$ for simple pole.

2). $\text{Res}(z=a) = \dfrac{\phi^{m-1}(a)}{(m-1)!}$

for pole of order m, if

$$f(z) = \frac{\phi(z)}{(z-a)^m}$$

3). $\text{Res}(z=a) = \dfrac{1}{2\pi i} \int_C f(z)\,dz$ for pole of any order.

4). $\text{Res}(z=\infty) = -\dfrac{1}{2\pi i} \int_C f(z)\,dz$,

$\text{Res}(z=\infty) = -z f(z)$ if limit exists.

5). $\text{Res}(z=\infty) =$ negative of the co-efficient of $1/z$ in the expansion of $f(z)$ in the nbd of $z=\infty$.

6). $\int_C f(z)\,dz = 2\pi i \sum_{r=1}^{n} \text{Res}(z=z_r) = 2\pi i \times$ (sum of the residues).

7). If $f(z) = \dfrac{\phi(z)}{\psi(z)}$ has a simple pole at $z=a$, then

$$\text{Res}(z=a) = \frac{\phi(a)}{\psi'(a)}$$

This formula is applied at those places, where $\psi(z)$ can not be factored. These rules are illustrated by the following examples;

$\text{cosec}\, z = \dfrac{1}{\sin z} = \dfrac{\phi(z)}{\psi(z)}$

Here $\phi(z) = 1$, $\psi(z) = \sin z$

$\sin z = 0 \Rightarrow \sin 0$, $\boxed{z=0}$ $\Rightarrow \text{Res}(z=0) = \dfrac{\phi(0)}{\psi'(0)}$

$\text{Res}(z=0) = \dfrac{1}{\cos z} = 1$

V.I

Problem:- Find the residue of

$$\frac{z^3}{(z-1)^4 (z-2)(z-3)} \quad \text{at } z = 1, 2, 3.$$

Solution:- Let $f(z) = \dfrac{z^3}{(z-1)^4 (z-2)(z-3)}$

Take $\phi(z) = \dfrac{z^3}{(z-2)(z-3)}$. Then $f(z) = \dfrac{\phi(z)}{(z-1)^4}$

$$\text{Res}(z=1) = \frac{\phi^{(3)}(1)}{3!} \quad \text{——— (I)}$$

Bracking $\phi(z)$ into partial fractions

$$\phi(z) = z + 5 - \frac{8}{z-2} + \frac{27}{z-3}$$

$$\therefore \quad \phi'(z) = 1 + \frac{8}{(z-2)^2} - \frac{27}{(z-3)^2}$$

$$\phi''(z) = \frac{16}{(z-2)^3} + \frac{54}{(z-3)^3}$$

$$\phi^{(3)}(z) = \frac{48}{(z-2)^4} - \frac{162}{(z-3)^4}$$

$$\therefore \phi^{(3)}(1) = 48 - \frac{162}{16} = \frac{303}{8}$$

Using this in (I), we get

$$\text{Res}(z=1) = \frac{303}{8 \times 6} = \frac{101}{16} \quad \underline{\text{Ans}}$$

$$\frac{z^3}{(z-2)(z-3)} = \frac{z^3}{z^2-5z+6}$$

$$z^2-5z+6 \,\big)\, z^3 \,\big(\, z+5$$

$$\cancel{z^3} - 5z^2 + 6z$$

$$5z^2 - 6z$$

$$\cancel{5z^2} - 25z + 30$$

$$19z - 30$$

$$\frac{z^3}{(z-2)(z-3)} = z+5 + \frac{19z-30}{(z-2)(}$$

Now, $\dfrac{19z-30}{(z-2)(z-3)} = \dfrac{A}{(z-2)} + \dfrac{B}{(z-3)}$

$\therefore 19z-30 = A(z-3) + B($

Here $A = -$
$B =$

$\therefore \dfrac{19z-30}{(z-2)(z-3)} =$

$$\text{Res}(z=2) = \lim_{z\to 2} (z-2) f(z) = \lim_{z\to 2} \frac{z^3}{(z-1)^4 (z-3)} = \frac{8}{1 \times (-1)} = -8 \ \underline{\text{Ans}}$$

$$\text{Res}(z=3) = \lim_{z\to 3} (z-3) f(z) = \lim_{z\to 3} \frac{z^3}{(z-1)^4 (z-2)} = \frac{27}{(3-1)^4 (3-2)} = \frac{27}{16} \ \underline{\text{Ans}}$$

Similar problem:- Find residue of $\frac{1}{(z^2+1)^3}$ at $z=i$

Solution:- $f(z) = \frac{1}{(z^2+1)^3} = \frac{\phi(z)}{(z-i)^3}$,

where $\phi(z) = \frac{1}{(z+i)^3}$

$\therefore \quad \phi'(z) = \frac{-3}{(z+i)^4}$

$\phi''(z) = \frac{12}{(z+i)^5}$

$\phi''(i) = \frac{12}{(i+i)^5} = \frac{12}{(2i)^5} = \frac{3}{8i}$

$$\boxed{Res\,(z=i) = \frac{\phi''(i)}{2!} = \frac{3}{16i}} \text{ Ans}$$

$z=i$ is a pole of order 3.

Problem:- Determine the order of poles and values of residues of the function

(i) $\operatorname{cosec} z$ (ii) $\frac{z+3}{z^2-2z}$

Solution (i) $f(z) = \operatorname{cosec} z = \frac{1}{\sin z}$

Poles are given by $\sin z = 0 = \sin 0$

$\therefore \; z=0$ is simple pole of $f(z)$.

By formula 7).

Write $f(z) = \frac{\phi(z)}{\psi(z)}$, then $\phi(z) = 1$, $\psi(z) = \sin z$

$Res\,(z=0) = \lim_{z\to 0} \frac{\phi(z)}{\psi'(z)} = \lim_{z\to 0} \frac{1}{\cos z} = \frac{1}{\cos(0)} = 1.$

$\therefore \; \boxed{Res\,(z=0) = 1}$

2nd method.

We know that

$z\operatorname{cosec} z = 1 + \frac{z^2}{6} + \frac{7z^4}{360} + \cdots$

$\Rightarrow \operatorname{cosec} z = \frac{1}{z} + \frac{z}{6} + \frac{7z^3}{360} + \cdots$

$Res\,(z=0) =$ coeff. of $1/z$

$= \underline{1}$

(ii) $f(z) = \dfrac{(z+3)}{z^2-2z} = \dfrac{z+3}{z(z-2)}$

Poles of $f(z)$ are $z(z-2)=0$

or, $z=0$, $z=2$ both are simple poles.

$$Res(z=0) = \lim_{z\to 0} (z-0)\, f(z)$$

$$= \lim_{z\to 0} \cancel{z}\left[\frac{(z+3)}{\cancel{z}(z-2)}\right] = \lim_{z\to 0} \frac{z+3}{(z-2)} = \frac{3}{-2} = -\frac{3}{2}$$

$\therefore \boxed{Res\,(z=0) = -\frac{3}{2}}$ Ans

$$Res\,(z=2) = \lim_{z\to 2} (z-2)\, f(z) = \lim_{z\to 2} \frac{\cancel{(z-2)}(z+3)}{z\,\cancel{(z-2)}} = \frac{5}{2}$$

$\therefore \boxed{Res\,(z=2) = 5/2}$ Ans

Problem :- Find the residues of $\dfrac{z+1}{z^2(z-3)}$

Solution :- Let $f(z) = \dfrac{z+1}{z^2(z-3)}$

Poles of $f(z)$ are given by

$z^2(z-3)=0$

$z=0$ is a pole of order 2

$z=3$ is a simple pole.

$$Res\,(z=3) = \lim_{z\to 3} (z-3)\, f(z) = \lim_{z\to 3} \cancel{(z-3)} \times \frac{(z+1)}{z^2\cancel{(z-3)}} = \frac{4}{9}$$

For $z=0$, $f(z) = \dfrac{\phi(z)}{z^2}$, $\phi(z) = \dfrac{z+1}{z-3}$

$$\phi'(z) = \frac{1.(z-3) - 1\,(z+1)}{(z-3)^2}$$

$$\therefore \phi'(0) = \frac{(0-3) - 1\,(0+1)}{(0-3)^2} = -\frac{4}{9}$$

$$Res\,(z=0) = \frac{\phi'(0)}{1!} = -\frac{4}{9}$$

$\therefore \boxed{Res\,(z=0) = -\frac{4}{9},\quad Res\,(z=3) = \frac{4}{9}}$ Ans

Problem:- Find the residue of $\frac{z^3}{z^2-1}$ at $z=\infty$.

Solution:- Let $f(z) = \frac{z^3}{z^2-1}$

Then $f(z) = \frac{z^3}{z^2}\left(1-\frac{1}{z^2}\right)^{-1}$

$$= z\left[1+\frac{1}{z^2}+\frac{1}{z^4}+\frac{1}{z^6}+\cdots\right]$$

or $f(z) = z+\frac{1}{z}+\frac{1}{z^3}+\frac{1}{z^5}+\cdots$

$\boxed{\text{Res}\,(z=\infty) = -(\text{co-efficient of } 1/z) = -(1) = -1}$ Ans

Note:- If $h(z) = f(z)\cdot g(z)$, where $f(z)$ and $g(z)$ are analytic in some nbd of $z=a$ and $f(z)$ & $g(z)$ have zero of order m & n respectively at $z=a$ then $h(z)$ has zero of order $\boxed{m+n}$ at $z=a$.

Proof:- $f(z) = (z-a)^m \phi_1(z)$

$g(z) = (z-a)^n \phi_2(z)$

$h(z) = f(z)\cdot g(z)$

$= (z-a)^{m+n} \phi_1(z)\cdot\phi_2(z)$

$\Rightarrow \boxed{h(z) = (z-a)^{m+n}\, \psi(z)}$ where $\psi(z) = \phi_1(z)\cdot\phi_2(z)$.

Lemma:- In any closed and bounded region a meromorphic function can have only finite number of zeros or poles or both.

Proof:- If it will be infinite, they have limit points and becomes essential singularities.

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Note:- If $f(z)$ is analytic on a domain D and limit point of zeros of $f(z) \in D$ then $f(z)$ is identically zero on D.

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Result:- If $f(z)$ is entire and satisfying the condition

(i) $f(z_1+z_2) = f(z_1)+f(z_2)$ $\forall$ $z_1, z_2 \in \mathbb{C}$

Then $\exists$ $c \in \mathbb{C}$ s.t. $f(z) = cz$

(ii) If $f(z_1\cdot z_2) = f(z_1)\cdot f(z_2)$ $\exists$ $c\in\mathbb{C}$ s.t. $f(z) = e^{cz}$.

Proof:- The above results true for the real.

$\Rightarrow$ By Identity theorem, the above results also true for complex plane.

(i) Let $G(z) = f(z) - cz$

$G(z) = 0 \quad \forall z \in \mathbb{R}$

$\Rightarrow G(z) = 0 \quad \forall z \in \mathbb{C}$

$\Rightarrow f(z) = cz \quad \forall z \in \mathbb{C}$. Hence proved

(ii) Ily Let $G(z) = f(z) - e^{cz}$

$G(0) = 0 \quad \forall z \in \mathbb{R}$

$\Rightarrow G(z) = 0 \quad \forall z \in \mathbb{C}$

~~$\Rightarrow G(z) = c(z)$~~

$\Rightarrow f(z) = e^{cz} \quad \forall z \in \mathbb{C}$. Hence proved.

V.V.V.I

Argument Theorem:- If f is analytic within and on a +ve orientated closed contour C and does not vanish at C and has finite no. of poles within C, then

$$\boxed{\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}\,dz = \boxed{N-P} \in \mathbb{Z}}$$

where N and P are no. of zeros and poles respectively counting multiplicity.

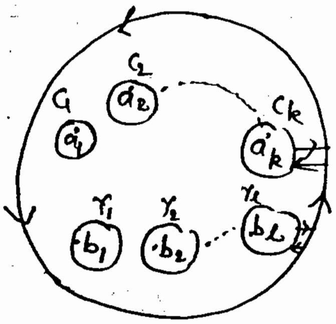
OR, If $f(z)$ is meromorphic in D and C is closed contour lying in D, and no zeros or poles of $f(z)$ lies on C and $h(z)$ is analytic within and on C then

(i) $$\boxed{\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}\,dz = N-P}$$

(ii) $$\boxed{\frac{1}{2\pi i}\int_C h(z)\frac{f'(z)}{f(z)}\,dz = \sum n_i\, h(a_i) - \sum p_i\, h(b_i)}$$

Where N is the no. of zeros counting multiplicity and P is the no. of poles counting multiplicity and a_i's and b_i's are zeros and poles of $f(z)$ of order n_i's and p_i's respectively.

$\boxed{\text{Proof}}$:- Let $f(z)$ has zeros of order n_i at a_i ; $i = 1$ to k
and $f(z)$ has poles of order p_i at b_i ; $i = 1$ to l.



(i) Now as a_i is a zeros of order n_i $\exists$ N_δ and $g_i(z)$ s.t.

$$f(z) = (z-a_i)^{n_i} g_i(z) \qquad g_i(a_i) \neq 0$$

$$g_i(0) \neq 0 \ \forall \ z \in N_\delta \ ; \ |z-a| < \delta.$$

Taking log and diff., we get

$$\frac{f'(z)}{f(z)} = \frac{n_i}{z-a_i} + \frac{g_i'(z)}{g_i(z)}$$

Let C_i be a closed contour in $N_\delta : |z-a| < \delta$ then

$$\frac{1}{2\pi i} \int_C \frac{f'(z)}{f(z)} dz = n_i \frac{1}{2\pi i} \int_{C_i} \frac{1}{(z-a_i)} dz + \underbrace{\frac{1}{2\pi i} \int_{C_i} \frac{g_i'(z)}{g_i(z)} dz}_{\to 0}$$

$$= n_i$$

(ii) As b_i is pole of order p_i
$\exists$ N_δ and $\phi_i(z)$ s.t.

$$f(z) = \frac{\phi_i(z)}{(z-b_i)^{p_i}} \qquad \phi_i(z) \neq 0 \ \forall \ z \in N_\delta$$

Taking log and diff. we get,

$$\frac{f'(z)}{f(z)} = -\frac{p_i}{z-b_i} + \frac{\phi_i'(z)}{\phi(z)}$$

$$\frac{1}{2\pi i}\int_{\gamma_i}\frac{f'(z)}{f(z)}dz = \frac{-p_i}{2\pi i}\int_{\gamma_i}\frac{1}{(z-b_i)}dz + \frac{1}{2\pi i}\int_{\gamma_i}\frac{\phi_i'(z)}{\phi_i(z)}dz$$

$$= -p_i$$

(iii) NOW $\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}dz = \sum_{i=1}^{k}\frac{1}{2\pi i}\int_{C_i}\frac{f'(z)}{f(z)}dz + \sum_{i=1}^{l}\frac{1}{2\pi i}\int_{\gamma_i}\frac{f'(z)}{f(z)}dz$

$$= \sum_{i=1}^{k} n_i - \sum_{i=1}^{l} p_i$$

$$= (n_1+n_2+\dots+n_k) - (p_1+p_2+\dots+p_l)$$

$$= N-P$$

i.e $\boxed{\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}dz = N-P}$

Problem :- If $f(z) = \frac{(z^2+1)^2}{(z^2+3z+2)^3}$ then find $\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}dz$

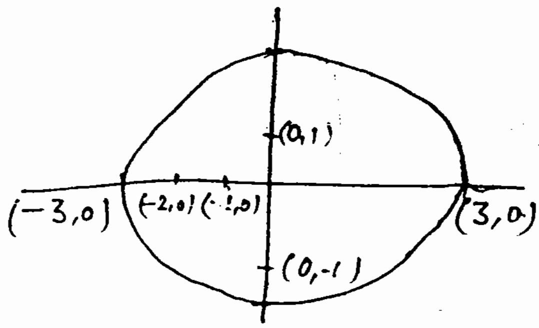
(i) $C: |z| = 3$ (ii) $C: |z| = 1/2$ (iii) $C: z = 3/2$

Solution :- No. of zeros $= 4$; $z^2+1=0 \Rightarrow z = \pm i$

No. of poles $= 6$; $z^2+3z+2=0 \Rightarrow z = -1, -2$.

(i) All points inside the $C: |z| = 3$ then

$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}dz = 4-6 = -2 \text{ Ans}$$



(ii) $\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)} = 0-0 = 0$ Ans

(iii) $\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)} = 4-3 = 1$ Ans

Problem :- Let $f(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$

$\alpha_1, \alpha_2, \alpha_3, \cdots \alpha_n$ are zeros

(i) $\frac{1}{2\pi i} \int_C z \frac{f'(z)}{f(z)} dz = ?$ (ii) $\frac{1}{2\pi i} \int_C z^2 \frac{f'(z)}{f(z)} dz = ?$

Solution :- (i) $\frac{1}{2\pi i} \int_C h(z) \frac{f'(z)}{f(z)} dz = \sum n_i \cdot h(a_i) - \sum p_i \cdot h(b_i)$

$= n_1 h(a_1) + n_2 h(a_2) + \cdots n_n h(a_n)$

$n_1 = n_2 = \cdots = n_n = 1 \quad = h(a_1) + h(a_2) + \cdots + h(a_n)$

$$\Rightarrow \frac{1}{2\pi i} \int_C z \frac{f'(z)}{f(z)} dz = \alpha_1 + \alpha_2 + \cdots + \alpha_n$$

$$= -\frac{a_{n-1}}{a_n}.$$

i.e. $$\boxed{\frac{1}{2\pi i} \int_C z \frac{f'(z)}{f(z)} dz = -\frac{a_{n-1}}{a_n}}$$ Ans

(ii) $\frac{1}{2\pi i} \int_C \frac{z^2 f'(z) dz}{f(z)} = \alpha_1^2 + \alpha_2^2 + \cdots + \alpha_n^2$

$$= (\alpha_1 + \alpha_2 + \cdots + \alpha_n)^2 - 2 \sum \alpha\beta$$

$$= \left[-\frac{a_{n-1}}{a_n}\right]^2 - 2\left(\frac{a_{n-2}}{a_n}\right)$$

$$= \frac{a_{n-1}^2 - 2a_{n-2}a_n}{a_n^2}$$

i.e. $$\boxed{\frac{1}{2\pi i} \int_C z^2 \frac{f'(z)}{f(z)} dz = \frac{a_{n-1}^2 - 2a_{n-2}a_n}{a_n^2}}$$ Ans

Problem:- $f(z) = z^3 - z + 1$, zeros are α, β, γ

find

(i) $\frac{1}{2\pi i}\int_C z^9 \frac{f'(z)}{f(z)}\,dz$

(ii) $\frac{1}{2\pi i}\int_C \frac{1}{z^2}\frac{f'(z)}{f(z)}\,dz$

Solution:-(i) $\frac{1}{2\pi i}\int_C z^9 \frac{f'(z)}{f(z)}\,dz = \sum n_i\, h(a_i) - \sum p_i\, h(b_i)$

$$= h(a_1) + h(a_2) + h(a_3) - 0$$

$$= \alpha^9 + \beta^9 + \gamma^9 = \sum \alpha^9$$

NOW. $z^3 - z + 1 = 0$

$z^3 = z - 1$

$\alpha^3 = \alpha - 1$, $\beta^3 = \beta - 1$, $\gamma^3 = \gamma - 1$

$\sum \alpha^3 = \sum \alpha - 3$

$\sum \alpha^3 = 0 - 3$ i.e $\sum \alpha^3 = -3$

$(\alpha^3)^3 = (\alpha - 1)^3 \Rightarrow \alpha^9 = \alpha^3 - 1 - 3\alpha^2 + 3\alpha$

$$\sum \alpha^9 = \sum \alpha^3 - 3 - 3\sum \alpha^2 + 3\sum \alpha$$

$$\sum \alpha^9 = -3 - 3 - 3\sum \alpha^2 + 3 \cdot 0$$

NOW, $\sum \alpha^2 = (\sum \alpha)^2 - 2\sum \alpha\beta$

$$= 0 - 2(-1) = 2$$

$$\sum \alpha^9 = -6 - 3(2) + 0$$

$$\boxed{\sum \alpha^9 = -12}$$

i.e. $$\boxed{\frac{1}{2\pi i}\int_C z^9 \frac{f'(z)}{f(z)}\,dz = -12}$$ Ans

(ii) $\frac{1}{2\pi i}\int_C \frac{1}{z^2}\frac{f'(z)}{f(z)}\,dz = \frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2}$

NOW. $\left(\frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2}\right)^2 = \frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2} + 2\left(\frac{1}{\alpha\beta} + \frac{1}{\beta\gamma} + \frac{1}{\gamma\alpha}\right)$

$$\left(\frac{\sum \alpha\beta}{\alpha\beta\gamma}\right)^2 = \sum \frac{1}{\alpha^2} + 2\left(\frac{\sum \alpha}{\alpha\beta\gamma}\right)$$

i.e. $\Sigma \frac{1}{\alpha^2} = \left(\frac{\Sigma\alpha\beta}{\alpha\beta\gamma}\right)^2 - 2\left(\frac{\Sigma\alpha}{\alpha\beta\gamma}\right)$

$= \left(\frac{-1}{-1}\right)^2 - 2(0) = 1.$

i.e. $\boxed{\frac{1}{2\pi i}\int_C \frac{1}{z^2}\frac{f'(z)}{f(z)}\,dz = 1}$ Ans.

Note:- $f(z) = \frac{p(z)}{q(z)} + \frac{c}{z-\alpha}$

(i) If $q(\alpha)=0$ (of order one)
then if $c = -\frac{p(\alpha)}{q'(\alpha)}$
then $f(z)$ is analytic at $\boxed{z=\alpha}$

(ii) If $q(\alpha)\neq 0$ then $c=0$ is only possible value for $f(z)$ is analytic at $z=\alpha$.

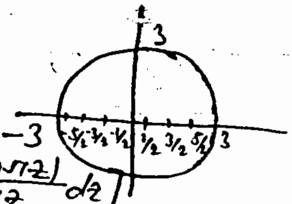
(iii) If $q(\alpha)=q'(\alpha)=0$
$\exists$ no 'c' for which $f(z)$ is analytic.

Note:- $\frac{p(z)}{q(z)} = \frac{A_1}{z-\alpha_1} + \frac{A_2}{z-\alpha_2} + \cdots + \frac{A_k}{z-\alpha_k}$

then $\boxed{A_i = \frac{p(\alpha_i)}{q'(\alpha_i)}}$

Doubt

Problem:- $\int_{|z|=3} \tan \pi z \, dz$: Solve it.



Solution:- $\int_{|z|=3} \tan \pi z \, dz = -\frac{2\pi i}{\pi}\left[\frac{1}{2\pi i}\int_{|z|=3} \frac{(-\pi \sin \pi z)}{\cos \pi z} dz\right]$

$$= -2i\left[\frac{1}{2\pi i}\int_{|z|=3} \frac{(-\pi \sin \pi z)}{(\cos \pi z)} dz\right]$$

Here $N=6$.

$P=0$

$\because \cos \pi z$ has no poles.

$$= -2i(N-P) = -2i(6-0)$$
$$= -12i$$

$$\boxed{\int_{|z|=3} \tan \pi z \, dz = -12i}$$ Ans

Problem:- Find $\int_{|z|=3} z \tan \pi z \, dz$

Sum of roots

Solution:- $\int_{|z|=3} z \tan \pi z \, dz = -2i\left(\frac{1}{2}+\frac{3}{2}+\frac{5}{2}-\frac{1}{2}-\frac{3}{2}-\frac{5}{2}\right)$

$$= 0$$

$$\therefore \boxed{\int_{|z|=3} z \tan \pi z \, dz = 0}$$ Ans

Note:- $$\boxed{\frac{1}{2\pi i}\int h(z) \frac{f'(z)}{f(z)} dz = \sum h(\alpha_i)}$$

Ques:- $f(z) = a_0 + a_1 z + \cdots\cdots + a_n z^n$

Find $\frac{1}{2\pi i}\int_{|z|=r} \sin z \cdot \frac{f'(z)}{f(z)} dz$.

Solution:- $\frac{1}{2\pi i}\int_{|z|=r} \sin z \frac{f'(z)}{f(z)} dz = \sum \sin(\alpha_i)$.

Problem:- $f(z) = \prod_{i=1}^{n} (z-i)$

$$\frac{1}{2\pi i}\int_{|z|=r} \sin z \frac{f'(z)}{f(z)} dz = \sum_{r=1}^{n} \sin r.$$

Problem :- For fixed +ve integer n and m;

compute $\int_{|z|=n} z^m \tan z \, dz$,

Where $|z|=n$ is the positively oriented circle.

Solution :- Here $f(z) = \cos z$ and $f'(z) = -\sin z$

The singularities of $f(z)$ are $(k+\frac{1}{2})\pi$; $k \in I$.

$$\int_{|z|=n} z^m \tan z \, dz = \int z^m \left(-\frac{\sin z}{\cos z}\right) dz$$

$$= -i \sum_{k=1}^{n-1} (k+\tfrac{1}{2})^m$$

V.V.I

Riemann Theorem :-

$$h(z) = \begin{cases} (z-b)^2 f(z) & ; z \neq b \\ 0 & ; z = b \end{cases}$$

and

(i) $f(z)$ is analytic in $D/\{b\}$.

(ii) $f(z)$ is bdd in $0 < |z-b| < \delta$; then $z=b$ is either regular point or removable singularity of $f(z)$.

Proof :- $h'(b) = \lim_{k \to 0} \frac{h(b+k) - h(b)}{k}$

$$= \lim_{k \to 0} \frac{(b+k-b)^2 f(b+k) - 0}{k}$$

$$= \lim_{k \to 0} k f(b+k) \qquad \because f \text{ is bdd in } 0<|z-b|<\delta$$

$\Rightarrow \quad h'(b) = 0$

$\Rightarrow$ $z=b$ is regular point of $h(z)$

Hence by using Taylor's series,

$$h(z) = \sum a_n (z-b)^n$$

$$= a_0 + a_1(z-b) + a_2(z-b)^2 + \text{-----}$$

$$= (z-b)^2 \left[a_2 + a_3(z-b) + \cdots \right]$$

$$h(z) = (z-b)^2 \left[a_2 + a_3(z-b) + \cdots \right]$$

$$h(z) = (z-b)^2 \sum_{n=0}^{\infty} C_n (z-b)^n \qquad \because a_0 = h(b) = 0$$

$$a_1 = \frac{h'(b)}{1!} = 0$$

$$h(z) = (z-b)^2 \sum_{n=0}^{\infty} C_n (z-b)^n$$

where $C_n = a_{n+2}$; $n = 0, 1, 2, \cdots$

$$(z-b)^2 f(z) = (z-b)^2 \sum C_n (z-b)^n \qquad \because z \neq b \text{ in limiting case.}$$

$$\boxed{f(z) = \sum C_n (z-b)^n}$$

$\Rightarrow$ This is the Taylor's series expansion of $f(z)$

$\Rightarrow$ $z = b$ is either regular point or removable singularity.

<u>Hence we can say that a function can be bounded in the nbd of isolated singularity.</u>

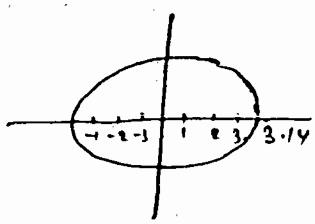
Problem:- Find $\int_C \frac{f'(z)}{f(z)} dz$; $C: |z| = \pi$

(a) $f(z) = \sin \pi z$

(b) $f(z) = \cos \pi z$

(c) $f(z) = \tan \pi z$

Solution:- (a) $\int_C \frac{f'(z)}{f(z)} dz = 2\pi i (N - P)$



Zeros of $\sin \pi z$ is given by

$$\pi z = n\pi$$

$$\Rightarrow z = n$$

$$z = 0, \pm 1, \pm 2, \pm 3.$$

There is no pole

$$\Rightarrow \int_C \frac{f'(z)}{f(z)} dz = 2\pi i (7 - 0)$$

$$= 14\pi i$$

$$\therefore \boxed{\int_{\pi C} \frac{f'(z)}{f(z)} dz = 14\pi i} \text{ Ans}$$

(b) Zeros of $\cos\pi z$ is given by

$$\pi z = (2n+1)\pi/2$$

$$\Rightarrow z = \frac{(2n+1)}{2} = n + \frac{1}{2}$$

$$\Rightarrow z = \pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{5}{2}$$

Also there is no pole i.e. $P = 0$

$$\Rightarrow \int_{|z|=\pi} \frac{f'(z)}{f(z)} = 2\pi i(6-0) = 12\pi i \text{ Ans}$$

(c) $f(z) = \tan\pi z = \dfrac{\sin\pi z}{\cos\pi z}$

No. of zeros are given by $\sin\pi z = 0$

$$\Rightarrow \pi z = n\pi$$

$$\Rightarrow z = n$$

$$\Rightarrow z = 0, \pm 1, \pm 2, \pm 3$$

No. of poles are given by

$$\cos\pi z = 0$$

$$\Rightarrow \pi z = (2n+1)\frac{\pi}{2}$$

$$\Rightarrow z = (2n+1)\frac{1}{2}$$

$$\Rightarrow z = n + \frac{1}{2}$$

$$\Rightarrow z = \pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{5}{2}$$

Hence $\displaystyle\int_{|z|=\pi} \frac{f'(z)}{f(z)}\, dz = 2\pi i(7-6) = 2\pi i$ Ans

Argument Principle:- If a function $f(z)$ is analytic within and on a positively oriented simple ~~path~~ closed contour C except at finite no. of poles inside C and number of zeros lie on the curve C then

$$\boxed{\frac{1}{2\pi} \Delta_C \arg f(z) = N - P}$$

Where $N =$ No. of zeros and $P =$ No. of poles

Here $\Delta_C \arg f(z)$ is variation of $\arg f(z)$ as z moves around C.

In particular:- If $f(z)$ is analytic within and on C and no poles on C then

$$\boxed{N = \frac{1}{2\pi} \Delta_C \arg f(z)}$$

Proof:-

$$\int_C \frac{d}{dz} [\log f(z)] dz = \int_C \frac{f'(z)}{f(z)} dz$$

$$\int_C \frac{f'(z)}{f(z)} dz = \Delta_C \left[\log |f(z)| + i \arg f(z) \right]$$

$$= 0 + \Delta_C \left[i \arg f(z) \right]$$

$$\Rightarrow 2\pi \cancel{i} (N - P) = \Delta_C \left[\cancel{i} \arg f(z) \right]$$

$$\Rightarrow (N - P) = \frac{1}{2\pi} \Delta_C \arg f(z)$$

i.e. $$\boxed{\frac{1}{2\pi} \Delta_C \arg f(z) = N - P}.$$

Rouche's Theorem :-

First form :- If $f(z)$ and $g(z)$ are analytic within and on a closed contour C and on C, $|f(z)| > |g(z)|$, then no. of zeros of $f(z)$ and $f(z)+g(z)$ are same within counting multiplicity.

Second form :- If $f(z)$ and $g(z)$ are analytic function within and on a closed contour C and $|g(z)-f(z)| < |f(z)|$ on C then $f(z)$ and $g(z)$ has same no. of zeros counting multiplicity within C.

Proof :- $|g(z)| < |f(z)|$ on C

Let no. of zeros of $f(z) = N_1$ &

" " " $(f+g)(z) = N_2$

Then by Argument principle,

$$N_1 = \frac{1}{2\pi} \Delta_C \arg f(z)$$

$$N_2 = \frac{1}{2\pi} \Delta_C \arg (f(z)+g(z))$$

$$\Rightarrow N_2 = \frac{1}{2\pi} \Delta_C \arg \left[f(z)\left(1 + \frac{g(z)}{f(z)}\right)\right]$$

$$\Rightarrow N_2 = \frac{1}{2\pi} \Delta_C \arg f(z) + \frac{1}{2\pi} \Delta_C \arg \left[1 + \frac{g(z)}{f(z)}\right]$$

$$N_2 - N_1 = \frac{1}{2\pi} \Delta_C \arg \left(1 + \frac{g(z)}{f(z)}\right)$$

Now, let $w = 1 + \frac{g(z)}{f(z)} \Rightarrow w - 1 = \frac{g(z)}{f(z)}$

But on C, $|f(z)| < |g(z)| \Rightarrow |w-1| < 1$

Now, as z moves on C, $w = f(z)$ moves around 1 without enclosing zero. Hence after one complete circle around C.

$$\Delta_C \arg w = 0 \Rightarrow \Delta_C \arg \left(1 + \frac{g(z)}{f(z)}\right) = 0$$

$$\Rightarrow N_2 - N_1 = 0 \Rightarrow \boxed{N_2 = N_1} \text{ Hence proved}$$

Problem:- Find the zeros of the given polynomial

$$f(z) = z^7 - 4z^3 + z - 1 \quad ; \quad |z| = 1.$$

Solution:- let $p(z) = -4z^3$ **When $|z| = 1$**

$q(z) = z^7 + z - 1$

$$\left|\frac{q(z)}{p(z)}\right| = \left|\frac{z^7 + z - 1}{-4z^3}\right|$$

$$< \frac{|z|^7 + |z| + 1}{4|z|^3} < \frac{3}{4} < 1 \qquad [\because |z| < 1]$$

$$\Rightarrow \quad |p(z)| > |q(z)|$$

Also, $p(z)$ has 3 zeros then by Rouche's theorem $p(z) + q(z) = f(z)$ also have 3 zeros.

When $|z| = 2$:-

$p(z) = z^7$

$q(z) = -4z^3 + z - 1$

$$\left|\frac{q(z)}{p(z)}\right| = \left|-\frac{4z^3 + z - 1}{z^7}\right|$$

$$< \frac{4|z|^3 + |z| + 1}{|z|^7}$$

$$= \frac{4(2)^3 + 2 + 1}{(2)^7} \qquad [\because |z| = 2]$$

$$= \frac{4 \times 8 + 2 + 1}{128} = \frac{35}{128}$$

$$\left|\frac{q(z)}{p(z)}\right| < \frac{35}{128} < 1$$

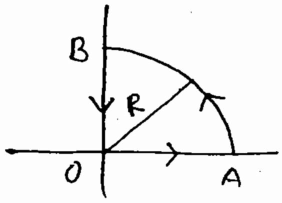
$$\Rightarrow \quad |p(z)| > |q(z)|$$

Also, $p(z)$ has 7 zeros then by Rouche's theorem $p(z) + q(z) = f(z)$ also have 7 zeros.

NET DEC 07

Problem:- $f(z) = 1 + z + z^4$

Locate all the zeros of $f(z)$ in argand plane (quadrant basis and axis basis).



Solution:- $f(z) = 1 + z + z^4$

When $y = 0$, $f(x) \neq 0$ for any $x \in [0, \infty]$

$$f(iy) = 1 + iy + y^4$$
$$= 1 + y^4 + iy$$

$f(iy) \neq 0$ for any $y \in [0, \infty]$

Thus $f(z)$ has no zero on OA and OB.

Change of argument along OA:- On OA, $y = 0$

$$f(z) = 1 + x + x^4 + i \cdot 0$$

$$\arg f(z) = \tan^{-1} \frac{0}{1 + x + x^4}$$

$$\Rightarrow \arg f(z) = 0$$

$\Delta_{OA} \arg f(z) = 0$ as $x \to \infty$

Along arc AB:-

$$f(z) = 1 + z + z^4$$

Let $z = Re^{i\theta}$

$$f(z) = 1 + Re^{i\theta} + R^4 e^{4i\theta}$$

$$\Rightarrow f(z) = R^4 e^{4i\theta} + Re^{i\theta} + 1$$

$$\Rightarrow f(z) = R^4 e^{4i\theta} \left[1 + \frac{1}{R^3} e^{-3i\theta} + \frac{1}{R^4} e^{-4i\theta}\right]$$

$f(z) \sim R' e^{i\phi}$ as $R \to \infty$ where $R' = R^4$, $\underline{\underline{\phi = 4\theta}}$

$$\Delta_{AB} \arg f(z) = \cancel{4}^{\,2} \cdot \frac{\pi}{\cancel{2}} = 2\pi$$

Along BO:-

$$f(z) = 1 + y^4 + iy$$

$$\arg f(z) = \tan^{-1} \frac{y}{1 + y^4}$$

When $y \to \infty$, $\arg f(z) = 0$

and when $y = 0$, $\arg f(z) = 0$.

Also $y > 0$ along BO

$\Rightarrow \Delta_{BO} \arg f(z) = 0$

Hence $\Delta_C \arg f(z) = 0 + 2\pi + 0 = 2\pi$

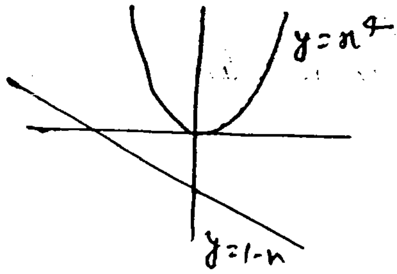
No. of zeros in first quadrant $= 1$ (By Argument principle)

$\frac{1}{2\pi} \times 2\pi = 1$

Hence there is one root in the 4th quadrant ($\because$ roots are complex)

($\because$ co-eff's of $f(z)$ are real)

When $x < 0$, $y = 0$; then there is no root of $f(z)$ on axis by graph $x^4 = 1 - x$.



Other two roots are also complex not purely imaginary.

They should lie in different quadrants (II & III).

Thus all the roots of $f(z)$ lie in different quadrants.

$\boxed{\text{Problem}}$:- $f(z) = \dfrac{(z^2+1)^2}{(z^2+3z+2)^3}$; $C: |z| = 3$

Evaluate $\int_C \frac{f'(z)}{f(z)} dz$

$\boxed{\text{Solution}}$:- $f(z) = \dfrac{(z^2+1)^2}{(z^2+3z+2)^3}$

Zeros of $f(z)$ are given by $z^2+1=0 \Rightarrow z = \pm i$

Total no. of zeros $= 2 \times 2 = 4$

Poles of $f(z)$ are given by $z^2+2z+z+2=0$

$\Rightarrow (z+2)(z+1) = 0$

$\Rightarrow z = -1, -2$

Total no. of poles $= 3 \times 2 = 6$

$\int_{|z|=3} \frac{f'(z)}{f(z)} dz = 2\pi i \times (4-6) = \underline{-4\pi i}$ Ans

Problem:- $f(z) = \dfrac{(z^2+1)^2}{(z^2+3z+2)^3}$; $C: |z| = 1.5$

Evaluate $\int_C \dfrac{f'(z)}{f(z)} dz$

Solution:- When $C: |z| = 1.5$

Poles are given by $z^2+3z+2=0$

$\Rightarrow (z+1)(z+2)=0$

$\Rightarrow z=-1,\ z \neq -2 \notin C: |z| = 1.5$

$$\frac{1}{2\pi i} \int_{|z|=1.5} \frac{f'(z)}{f(z)} dz = [2\times 2 - 3\times 1]$$

$$\Rightarrow \int_{|z|=1.5} \frac{f'(z)}{f(z)} dz = 2\pi i \times (4-3)$$

$$\Rightarrow \int_{|z|=1.5} \frac{f'(z)}{f(z)} dz = 2\pi i \quad \text{Ans}$$

Interview

Problem:- Find all the entire functions satisfying

$f(i + \frac{1}{n}) = 0$

Solution:- Limit point of zeros $i + \frac{1}{n} = i$

$\Rightarrow$ Singleton set.

Problem:- Find all the entire functions satisfying

$f(z) = 0$; $z = \{0\} \cup \left\{\frac{1}{4n+7} ; n \in \mathbb{N}\right\}$

Solution:- Limit point of zeros $\in \{0\} \cup \left\{\frac{1}{4n+7} ; n \in \mathbb{N}\right\}$

$\Rightarrow \boxed{f(z) \equiv 0}$

Imp objective

Hurwitz Theorem :- Let $\langle f_n(z)\rangle$ be a sequence of analytic function define on a domain D and $f_n(z)$ is not equal to zero $\forall\, z \in D$ and $n \in \mathbb{N}$. Further let $f_n(z)$ converges uniformly to $f(z)$ on every compact subset of D then $f(z)$ is identically zero or $f(z)$ has no zeros in D.

V.V.I.

Mean Value Property :- A complex valued function $f(z)$ is defined on a domain D is said to have mean value property if for every $a \in D$ and $r > 0$ s.t. $N_r(a) = |z-a| \le r$ is contained in D then

$$\boxed{f(a) = \frac{1}{2\pi}\int_0^{2\pi} f(a + re^{it})\,dt}$$

$f(a)$ is called mean value or average value of $f(z)$ on the circle $|z-a| \le r$.

Property :-

1). If $f(z)$ is analytic in D then $f(z)$ has mean value property.

2). If $f(z) = u + iv$ is analytic then component function i.e u and v have mean value property in that domain D and define as

$$\boxed{u(a) = \frac{1}{2\pi}\int_0^{2\pi} u(a + re^{it})\,dt}$$

and

$$\boxed{v(a) = \frac{1}{2\pi}\int_0^{2\pi} v(a + re^{it})\,dt}$$

V.V.I.

Theorem 1 :- If $f(z)$ is non-constant continuous complex valued function define in a domain D and satisfy the mean value property then $|f(z)|$ does not attain its maximum value in D unless it is constant.

Theorem 2 :- If $f(z)$ is continuous complex valued function of boundary of D (i.e. $\bar{D}$) and have mean value property then $|f(z)|$ attains its maximum value on the boundary of D i.e. $\bar{D}$.

Objective

Maximum Modulus Principle :- If $f(z)$ is analytic in a domain D then $|f(z)|$ does not attain its maximum value in D unless it is constant.

Note :- If $f(z)$ is analytic in D and $\exists\ a \in D$ s.t. $|f(z)| \le |f(a)|\ \forall z \in$ then $f(z)$ is constant.

e.g. :- If $f(z)$ is analytic in $|z|<5$

$|f(z)| \le 1\ \forall\ z \in D : |z|<5$

& $f(0) = 1 \Rightarrow$ It attains $\max^m$ value

$\Rightarrow f(z)$ is constant by Maximum Modulus Principle.

Objective.

Maximum Modulus Theorem :- If $f(z)$ is analytic in D and continuous on the boundary of D i.e $\bar{D}$ then maximum of $|f(z)|$, which is always attained or reached somewhere on the boundary of D but never in the interior.

Application :- Let u be a harmonic function in D and continuous on the boundary of D then $\max. |u(x,y)|$ occurs on the boundary.

Since u is harmonic function then $\exists\ v$ s.t.

$$f(z) = u + iv$$
$$g(z) = e^{f(z)} = e^{u+iv}$$
$$|g(z)| = |e^{u+iv}| = e^{|u|}$$

$\max. |g(z)|$ is attained on the boundary of D

$\Rightarrow \max. |e^u|$ " " " " " " " .

$\Rightarrow \max |u|$ " " " " " " "

~~Minimum Modulus Principle :- If $f(z)$ is analytic in D and continuous on $\bar{D}$ s.t. $f(z) \ne 0$ for $z \in D$ then the minimum of $|f(z)|$ is attained somewhe~~

Objective

Minimum Modulus Principle :- If $f(z)$ ~~is analytic in D s.t. f(z)~~ is never zeros in D then $|f(z)|$ does not attain its minimum value in D unless it is constant.

OR, If $f(z)$ is analytic in D and does not vanish for any $z \in D$ then $|f(z)|$ does not attain its minimum value in D unless it is constant.

Minimum Modulus Theorem :- If $f(z)$ is non-vanishing analytic function in D and continuous on the boundary of D then minimum $|f(z)|$, which is always attained or reached somewhere on the boundary of D and never in the interior.

Result :- If $f(z)$ is a non-constant analytic function defined in $D: \{|z|<r\}$ and continuous on the boundary of D s.t. $|f(z)|>m$ on ∂D. If $|f(0)|<m$ then $\exists$ at least one zero of $f(z)$ in $D: |z|<r$.

Solution :- Suppose $f(z) \neq 0$ in $|z|<r$. Since $f(z)$ is non-constant analytic function in $|z|<r$ and continuous on $|z|=r$, it follows that the minimum of $|f(z)|$ occurs somewhere on $|z|=r$. But this contradicts the hypothesis $|f(0)|<m$ and $|f(z)|>m$ on $|z|=r$.

Hence $f(z)$ vanishes for at least one point in $|z|<r$.

Q.15 :- If $f(z)$ is analytic in D s.t. maximum $|f(z)|$ is attained somewhere in D then $f(z)$ is constant.

Problem :- If $f(z) = z^2+2$, then determine the minimum value of $|f(z)|$ over the closed region $|z| \leq 1$.

Solution :- Since the zeros of $f(z)$, namely $z = \pm\sqrt{2}i$ are outside the disc $|z| \leq 1$, it follows from the Minimum modulus theorem that the minimum of $|f(z)|$ occurs on the boundary $|z|=1$

Hence our problem is actually not a problem of two independent variables x and y raising over $|z|=1$, but rather a problem involving a single variable, the argument θ of a variable raising over $|z|=1$, setting $x=\cos\theta$, $y=\sin\theta$, we note at once that

$$|f(z)| = |z^2+2|$$
$$= \sqrt{(x^2-y^2+2)^2+(2xy)^2}$$
$$= \sqrt{(\cos 2\theta+2)^2+\sin^2 2\theta}$$
$$|f(z)| = \sqrt{4\cos 2\theta+5}$$

The extreme values of $|f(z)|$ are thus determined by the condition

$$\frac{d|f(z)|}{d\theta} = \frac{-\cancel{8}^{4}\sin 2\theta}{\cancel{2}\sqrt{4\cos 2\theta+5}} = 0$$

Put $\frac{d|f(z)|}{d\theta}=0$

$\sin 2\theta = 0 = \sin n\pi$

$\Rightarrow 2\theta = 0, \pi, 2\pi, 3\pi$

$\Rightarrow \theta = 0, \pi/2, \pi, 3\pi/2$

$$\frac{d^2|f(z)|}{d\theta^2} = -4\left[\frac{10\cos 2\theta+8}{(5+4\cos 2\theta)^2}\right]$$

$z=1$ at $\theta=0$, $|f(z)|=3$
$z=i$ at $\theta=\pi/2$, $|f(z)|=1$
$z=-1$ at $\theta=\pi$, $|f(z)|=3$
$z=-i$ at $\theta=3\pi/2$, $|f(z)|=1$

At $\theta=0, \pi$

$$\frac{d^2|f(z)|}{d\theta^2} = -ve = -\frac{20}{9}$$

At $\theta = \pi/2, 3\pi/2$

$$\frac{d^2|f(z)|}{d\theta^2} = +ve = 8$$

i.e. $\min|f(z)| = 1$ at $\theta = \pi/2, 3\pi/2$

$z = 1, -1 \rightarrow \max$
$z = i, -i \rightarrow \min$

Hence $\theta=0, \pi/2, \pi, 3\pi/2$ are the values where the extreme occurs. The values $\pi/2$, $3\pi/2$ corresponds to the minimum value 1. Also the value $0, \pi$ corresponds to $\max^m$ value 3.

Problem:- If $f(x)$ is differentiable and $f(x) < 5$ and $f(1) = 5$ then find $f'(1)$.

Solution:- At $x = 1$,
$f(x)$ attains global maximum value.
$\Rightarrow f'(1) = 0$.

Problem:- If $f(z)$ is analytic in D and continuous on the boundary of $D: |z-1| = 2$.
If $f(1) = 1$ and $|f(z)| \le 1$ then $f(i) = ?$

Solution:- By C_{15} function $f(z)$ is constant
Then $f(i) = 1$.

Casorati Weierstrass Theorem:- If $f(z)$ is analytic in $D/\{a\}$ where 'a' is essential singularity of $f(z)$. Let w_0 be any complex number then $\exists$ a sequence $\langle z_n \rangle$ converging to 'a'. s.t. $\langle f(z_n) \rangle \rightarrow w_0$ i.e. a function can get close to any complex number in the nbd of essential singularity.

Proof:- We prove the theorem by contradiction that means there exists a complex number w_0 s.t. for given $\varepsilon > 0$, we have
$\forall z \in 0 < |z-a| < \delta$
$\Rightarrow |f(z) - w_0| \geq \varepsilon$
Let $z_n \rightarrow a$ but $f(z_n) \not\rightarrow w_0$ —— (1)
Let us suppose that for each δ, $0 < |z-a| < \delta$, we take

$$|f(z) - w_0| > \varepsilon$$

~~Let us suppose that for each δ,~~

The function $f(z) - w_0$ is analytic in $D/\{a\}$ and does not vanish due to (1).

So, $\frac{1}{f(z) - w_0}$ is analytic in $D/\{a\}$ and our assumption

$\frac{1}{f(z) - w_0} \le \varepsilon$. This implies $\frac{1}{f(z) - w_0}$ is bdd in $0 < |z-a| < \delta$.

Let us define a function

$$g(z) = \frac{1}{f(z) - w_0} \quad \text{is analytic in } D/\{a\}.$$

$$|g(z)| < \frac{1}{\varepsilon} \Rightarrow g(z) \text{ is bdd in } 0 < |z-a| < \delta$$

$\therefore$ By Riemann theorem the function $g(z)$ is ~~entire~~ either analytic at 'a' or 'a' is removable singularity.

When 'a' is removable singularity then redefine the function

$$f(z) = \frac{1}{g(z)} + w_0$$

Let $g(a)$ be defined so that $g(z)$ is analytic at 'a'.
Since $f(z)$ cannot be constant function, neither can $g(z)$; in view of the Taylor's series for $g(z)$ at 'a' either $g(a) \neq 0$ or $g(z)$ has a zero of some finite order at 'a'. Consequence its reciprocal

$$\frac{1}{g(z)} = f(z) - w_0$$ is either analytic at 'a' and has a pole there. But contradicts the hypothesis that $z=a$ is an essential singularity.

Thus condition must be satisfied at some point in the given deleted nbd.

Hence proved.

Problem:- $f(z) = f(kz)$, $|k| > 1$ $\forall$ $z \in \mathbb{C}$ and f is entire

$\Rightarrow f(z)$ is constant.

Solution:- $f(z) = f(kz)$

$$f\left(\frac{z}{k}\right) = f(z)$$

$$\Rightarrow f(z) = f\left(\frac{z}{k}\right)$$

$$= f\left(\frac{z}{k^2}\right)$$

$$\vdots$$

$$= f\left(\frac{z}{k^n}\right)$$

At any point z, $f(z)$ can be taken as close as possible to $f(0)$

$\Rightarrow$ $f(z)$ is bdd.

$\Rightarrow$ $f(z)$ is constant. (By Liouville's theorem).

OR, For $z_1 \neq z_2$

$$f(z_1) = f\left(\frac{z_1}{k^n}\right)$$

$$|f(z_1) - f(0)| < \varepsilon/2.$$

Similarly

$$|f(z_2) - f(0)| < \varepsilon/2$$

$$|f(z_1) - f(z_2)| = |f(z_1) - f(0) + f(0) - f(z_2)|$$

$$\leq |f(z_1) - f(0)| + |f(z_2) - f(0)|$$

$$< \varepsilon/2 + \varepsilon/2$$

$\Rightarrow |f(z_1) - f(z_2)| < \varepsilon$

$\Rightarrow f(z_1) = f(z_2)$

$\Rightarrow f(z)$ is constant.

Problem:- $f(z)$ is analytic in D and continuous on the boundary of D: $|z-1|=2$

If $f(1)=1$ and $|f(z)|\le 1$ then $f(i)=?$

Soln:- By G_5, $f(z)$ is constant. Then $f(i)=1$.

Problem:- $f(z) = |z|^2 + \frac{1}{z}$

Solution:- $f(z) = z\bar{z} + \frac{1}{z}$

$$\frac{\partial f}{\partial \bar{z}} = z$$

$\Rightarrow$ $f(z)$ is nowhere differentiable except at $z=0$ & function is nowhere analytic.

Imp.

Result:- If $f(z)$ is analytic and $g(z)$ is meromorphic with at least one pole then $f(z)+g(z)$ is meromorphic with at least one pole.

Imp.

Result:- If $f(z)$ fails to be analytic and $g(z)$ is meromorphic then it is possible that $f(z)+g(z)$ has no singularity.

e.g:- $\phi(z) = \underbrace{\operatorname{Re} z}_{f(z)} + \underbrace{\frac{1}{z}}_{g(z)}$ $\qquad \operatorname{Re} z = \frac{z+\bar{z}}{2}$

$$\phi(z) = \frac{z+\bar{z}}{2} + \frac{1}{z}$$

$$\frac{\partial \phi}{\partial \bar{z}} = \frac{1}{2} \neq 0$$

$\Rightarrow$ Nowhere analytic and has no singularity.

Problem:- If $f(z)$ and $g(z)$ are analytic and non-zero functions s.t $f(z) = u+iv$ and $g(z) = v+iu$ then

(i) $f'(z)=0$

(ii) $f(z)$ is conformal.

(iii) $f(z) = k\,g(z)$, $k\in\mathbb{R}$

(iv) $f(z)$ is 1-1 or univalent.

Solution :- Since $f(z)$ and $g(z)$ are both ~~analytic functions~~.
This implies u and v both are harmonic conjugate of each other and hence u & v are constant.

Let $u = \alpha$, $v = \beta$

$$\Rightarrow f(z) = u + iv = \alpha + i\beta = c_1$$
$$\& \ g(z) = \cancel{v+iu} = \beta + i\alpha = c_2$$

Where c_1 and c_2 are some complex numbers.

$$f(z) = c_2 \underbrace{(c_2^{-1} c_1)}_{k}$$
$$\Rightarrow f(z) = k\, g(z).$$

V.V. Imp

Schwarz Lemma :- If $f(z)$ is analytic in D, $|z|<1$, with the zeros of order n at the origin and $|f(z)|<1 \ \forall z$ in $|z|<1$ then $|f(z)| \le |z|^n \ \forall |z| \le 1$ and $|f^n(0)| \le n!$

Further if $|f(z)| = |z|^n$ for some $z \in 0<|z|<1$ then $f(z) = cz^n$ where $|c|=1$.

Note :- If $f(z)$ is analytic in D and set of zeros of $f(z)$ has limit point in D then $f(z)$ is identically zero.

Identity Theorem :- If $f(z)$ and $g(z)$ are analytic in D s.t. $f(z) = g(z) \ \forall z \in S$ and if S has limit point in D then $f(z) = g(z) \ \forall z \in D$.

Imp

Application :- If $f(z)$ and $g(z)$ are entire functions s.t. $f(1/n) = g(1/n) \ \forall n \in \mathbb{N}$ then show that $f(z) = g(z) \ \forall z \in \mathbb{C}$.

Solution :- Contruct $h(z) = f(z) - g(z)$
$$h(1/n) = 0 \quad \forall n \in \mathbb{N}$$

Now, $S = \{1, 1/2, 1/3, \ldots\}$ be the set of zeros of $h(z)$ i.e. l.p of zeros i.e. $\boxed{h(z) \equiv 0}$

then $f(z) = g(z) \ \forall z \in \mathbb{C}$ H.P

Note: (1). A non-constant entire function may have infinite zeros but the limit point of the set of all zeros can never lies in the finite part of the complex plane. If it lies in the finite part of complex plane then function becomes identically zero.

Note (2). If $f(z)$ and $g(z)$ are entire functions and gets same value on the curve then they are identically on C (constant

$$f(z) = \sin^2 z + \cos^2 z - 1$$

[Identity theorem]

On the real line

$$f(z) = \sin^2 x + \cos^2 x - 1 = 0$$

ZGTR

$$f(z) = \sin^2 z + \cos^2 z - 1 = 0$$

i.e $\boxed{\sin^2 z + \cos^2 z = 1}$

Problem:- $f(1/n) = 1/n^2 = f(-1/n)$. Find $f(z)$?

Solution:- Now construct $g(z) = f(z) - z^2$

$$g(1/n) = f(1/n) - 1/n^2 = 0$$

$$g(-1/n) = f(-1/n) - 1/n^2 = 0$$

Satisfy the analytic condition

i.e. $g(z) \equiv 0$.

then $\boxed{f(z) = z^2}$ Ans

Problem:- $f(1/n) = 1/n^3 = f(-1/n)$, find $f(z)$?

Solution:- Now construct $g(z) = f(z) - z^3$

$$g(1/n) = f(1/n) - 1/n^3 = 0$$

$$g(-1/n) = f(-1/n) - 1/n^3 = -1/n^3 - 1/n^3 = -2/n^3$$

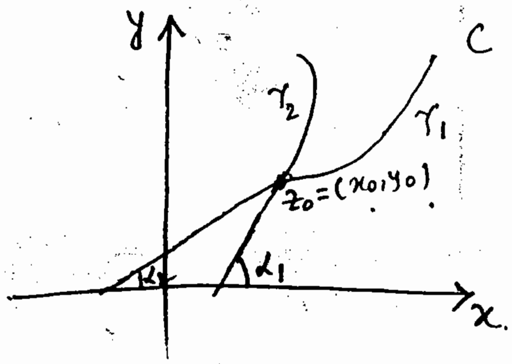
i.e. $\boxed{f(z) = -z^3}$ or $\boxed{f(z) = z^3}$

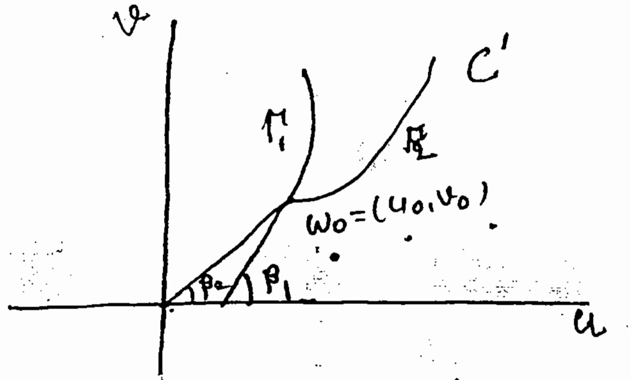
not analytic

UNIT-4

CONFORMAL MAPPING

Conformal Mapping / Conformality :- A function $w=f(z)$ is said to be conformal at z_0 if curve in the z-plane passing through z_0 and image curve in w-plane passing through $f(z_0)$ preserves the angle in magnitude & sense of rotation of angle.





Isogonal :- If the angle between two curves is preserved the sense of magnitude only then we say the transformation function $w=f(z)$ is isogonal at that point.

Problem :- Find the image of the region $y=x$ under the map $f(z)=z^2$.

Solution :- $f(z)=z^2$

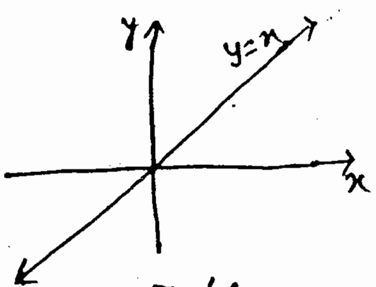
$= (x+iy)^2$

$\Rightarrow u+iv = x^2-y^2+i(2xy)$

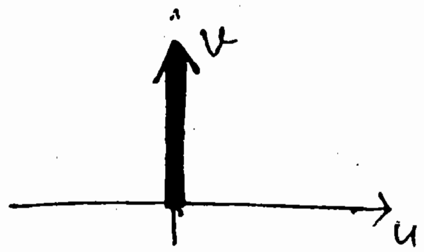
$u=x^2-y^2, \quad v=2xy$

$u=0$ [in $y=x$]

& $v=2x^2>0$



z-plane



f(z) = w-plane

Problem:- Find the image of the region $y=x$ under the map $w=\bar{z}$.

Solution: $f(z)=\bar{z}$

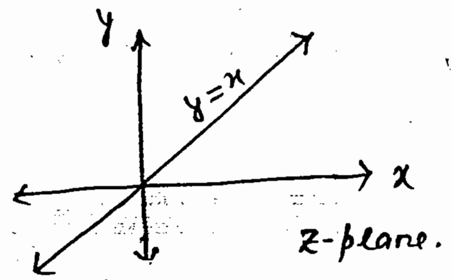
$u+iv = x-iy$

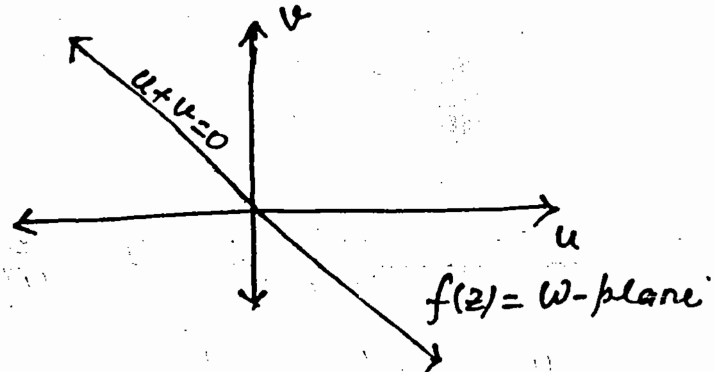
$u=x,\ v=-y$

$v=-x\ [\because\ y=x]$

$\Rightarrow v=-u$

$\Rightarrow u+v=0$

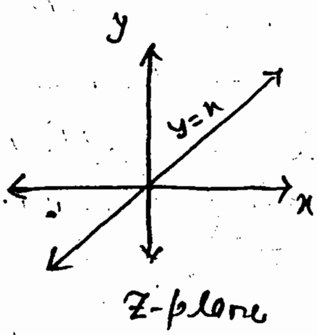


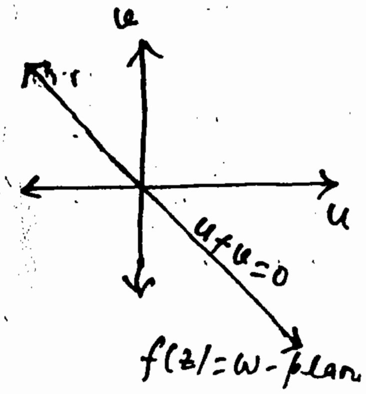


Problem:- Find the image of the region $y=x$ under the map $w=\frac{1}{z}$.

Solution:- $f(z)=\frac{1}{z} = \frac{\bar{z}}{|z|^2}$

$$= \frac{x-iy}{x^2+y^2}$$





$$f(z)= \frac{x}{x^2+y^2} + i\left(-\frac{y}{x^2+y^2}\right)$$

$$u=\frac{x}{x^2+y^2},\quad v=\frac{-y}{x^2+y^2}$$

$$u=\frac{x}{x^2+y^2} = \frac{x}{x^2+x^2} = \frac{1}{2x} \quad [\because y=x]$$

$$v=\frac{-y}{x^2+y^2} = \frac{-x}{x^2+y^2} = -\frac{1}{2x} \quad [\because y=x]$$

$\Rightarrow v=-u$

$\Rightarrow u+v=0$.

Problem:- The map $w = \frac{1}{z}$ transforms circles and lines into circle and lines.

Proof:- Let a, b, c, d be real numbers such that $b^2+c^2 > 4ad$. Then $\boxed{a(x^2+y^2)+bx+cy+d=0}$ —— ① represents the equation of circle ~~repres~~ in z-plane.

$$f(z) = \frac{1}{z} = \frac{\bar{z}}{|z|^2} = \frac{x-iy}{x^2+y^2}$$

$$u = \frac{x}{x^2+y^2}, \quad v = \frac{-y}{x^2+y^2}$$

$$u^2+v^2 = \frac{1}{x^2+y^2} \Rightarrow x^2+y^2 = \frac{1}{u^2+v^2}$$

$$x = \frac{u}{u^2+v^2}, \quad y = \frac{v}{u^2+v^2}$$

Put in ①, we get

$$\boxed{d(u^2+v^2)+bu-cv+a=0} \quad \text{—— (2)}$$

It conclude the following:

(A) A circle not passing through origin $(a \neq 0, d \neq 0)$ in the z-plane is mapped on a circle in w-plane not passing through the origin.

(B) A circle passing through the origin $(a \neq 0, d = 0)$ in the z-plane is mapped on a line in the w-plane not passing through the origin.

(C) A line not passing through the origin $(a = 0, d \neq 0)$ in the z-plane is mapped on a circle in the w-plane passing through the origin.

(D) A line passing through origin $(a = 0, d = 0)$ in the z-plane is mapped on a line in the w-plane passing through the origin.

Problem:- Find the image of infinite strip $0<y<\frac{1}{2c}$ under the map $w=\frac{1}{z}$.

Solution:- $f(z)=\frac{1}{z}=\frac{\bar{z}}{|z|^2}$

$$u+iv=\frac{x-iy}{x^2+y^2}$$

$$u=\frac{x}{x^2+y^2}\ ,\quad v=\frac{-y}{x^2+y^2}$$

$$y>0$$

$$\Rightarrow v<0$$

$$u^2+v^2=\frac{1}{x^2+y^2}$$

$$x^2+y^2=\frac{1}{u^2+v^2}$$

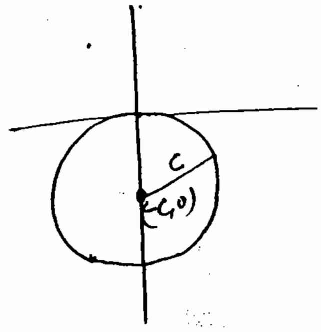
$$\frac{1}{x^2+y^2}=\frac{-v}{y}$$

$$\frac{-v}{u^2+v^2}<\frac{1}{2c}$$

$$\Rightarrow u^2+v^2>-2cv$$

$$\Rightarrow u^2+v^2+2cv>0$$

i.e. the region outside the circle.



Necessary and sufficient condition for conformality:-

If $w=f(z)$ is analytic in the nbd of z_0 then it is conformal there at provided $f'(z_0)\neq 0$

i.e If $w=f(z)$ is conformal at z_0 if z_0 is <u>regular point</u> and $f'(z_0)\neq 0$.

Problem:- Which of the following function is <u>not</u> conformal at $z=0$.

(i) e^z (ii) z^2+2z (iii) $\sin z$ (iv) e^{z^2}

Solution:- $f(z)=e^{z^2}\Rightarrow f'(z)=2ze^{z^2}\Rightarrow f'(0)=0$

$\Rightarrow e^{z^2}$ is not conformal at $z=0$

Note:- If $f(z)$ is conformal at z_0 then $\exists$ a ~~nbd of z_0~~ in which $f(z)$ is univalent i.e. one-one.

Angle of rotation:- Let $\gamma \equiv z(t)$ be a smooth curve given by $z(t) = x(t) + iy(t)$; $a \le t \le b$.

The curve $z(t)$ is smooth if $x'(t)$ and $y'(t)$ are continuous and $z'(t) = x'(t) + iy'(t) \neq 0$ $\forall$ $t \in (a,b)$.

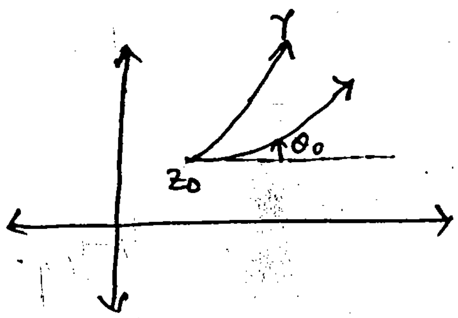
It is to be noted that increasing values of the parameter t will give the positive direction of the curve γ.

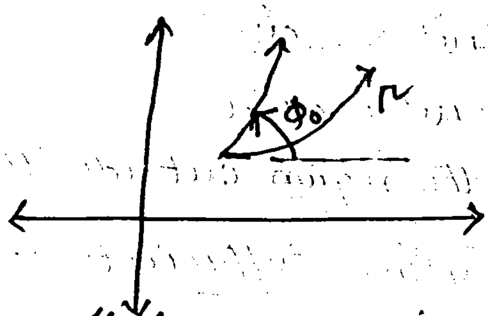
Let $f(z)$ be a function on γ. Then

$w = f(z) = f(z(t))$; $a \le t \le b$ is the parametric representation of the image of γ under the map $w = f(z)$.

Suppose γ passes through z_0. Assume that f is analytic at z_0 and $f'(z_0) \neq 0$. Then by the chain rule

$$w_0'(t_0) = f'(z_0(t))\, z_0'(t) \; ; \; t_0 \in [a,b] \;\&\; z_0 = z_0(t_0).$$





Since $f'(z_0) \neq 0$, it follows that

$$\arg w'(t_0) = \arg f'[z(t_0)] + \arg z'(t_0)$$

$$\boxed{\phi_0 = \alpha + \theta_0}$$

Where θ_0 is the angle of inclination of the directed tangent to γ at z_0 and is a value of $\arg z'(t_0)$.

Also ϕ_0 is the angle of inclination of the directed tangent to Γ (image curve) whereas the rotation of γ is a curve of $\arg f'(z_0)$, which we have denoted by α.

OR,
$$w \longrightarrow \Gamma$$
$$z \longrightarrow \gamma$$

$$z_0 = \gamma(t_0) = x(t) + iy(t)$$
$$w_0 = \Gamma(t_0) = u(t) + iv(t)$$
$$\arg \gamma'(t_0) = \tan^{-1}\left(\frac{y'}{x'}\right) = \tan^{-1}\left(\frac{dy}{dx}\right)_{at\,(x_0, y_0)} = \alpha$$
$$\Gamma'(t_0) = u'(t) + iv'(t)$$
$$\arg \Gamma'(t_0) = \tan^{-1}\left(\frac{v'}{u'}\right) = \tan^{-1}\left(\frac{dv}{du}\right)_{at\,(u_0, v_0)} = \beta$$

$$f(\gamma(t))' = f'(\gamma(t))\,\gamma'(t)$$
$$\arg (f(\gamma(t)))' = \arg(f'(\gamma(t))) + \arg(\gamma'(t))$$

$\boxed{\beta = \theta + \alpha}$ where θ = angle of rotation.

NOTE:- The angle of rotation is a property of function at a point and implies any curve passing through this point in z-plane will change its direction by this angle in w-plane.

Problem:- $f(z) = w = z^2$

C: $x - y = 1$; $z_0 = 2 + i$

$\theta = ?$

Solution:- $f(z) = z^2$

$f'(z) = 2z$

$f'(z_0) = 2z_0 = 2(2+i) = 4 + 2i$

$$\arg(f'(z_0)) = \tan^{-1}\left(\frac{2}{4}\right) = \tan^{-1}(1/2) = \theta$$

$\boxed{\theta = \tan^{-1}(1/2)}$ Ans

$\boxed{\text{IInd method}}$:-

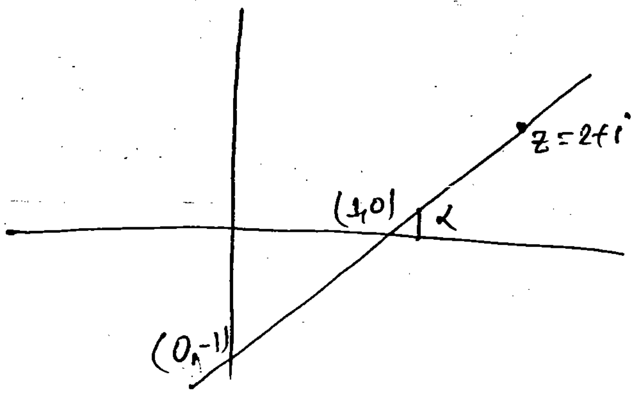
$y = x - 1$

$\Rightarrow \dfrac{dy}{dx} = 1$

$\Rightarrow m = \tan\alpha = 1$

$\alpha = \tan^{-1} 1 = \pi/4$

i.e $\boxed{\alpha = \dfrac{\pi}{4}}$



$f(z) = z^2 = (x+iy)^2 = x^2 - y^2 + 2ixy$

$u = x^2 - y^2, \quad v = 2xy$

$v = 2x(x-1) = 2x^2 - 2x$

$u = x^2 - (x-1)^2 = 2x - 1 \Rightarrow x = \dfrac{u+1}{2}$

$v = 2\left(\dfrac{u+1}{2}\right)^2 - 2\left(\dfrac{u+1}{2}\right) = \dfrac{1}{2}(u^2 - 1)$

$w_0 = z_0^2 = (2+i)^2 = 4 - 1 + 4i = 3 + 4i$

$u^2 = 2v + 1$

$= 2(v + 1/2)$

$\beta = \tan^{-1}\left(\dfrac{dv}{du}\right)_{at (3,4)} = \tan^{-1}(u)_{at (3,4)}$

$\Rightarrow \boxed{\beta = \tan^{-1} 3}$

$\beta = \theta + \alpha$

$\Rightarrow \theta = \beta - \alpha$

$= \tan^{-1} 3 - \tan^{-1}$

$= \tan^{-1}\left(\dfrac{3-1}{3+1}\right) = \tan^{-1}\left(\dfrac{1}{2}\right)$

$\Rightarrow \boxed{\theta = \tan^{-1}(1/2)}$ Ans

Formula:- $\alpha = \tan^{-1}\left(\frac{dy}{dx}\right)_{at\,(x_0,y_0)}$, $\beta = \tan^{-1}\left(\frac{dv}{du}\right)_{at\,(u_0,v_0)}$

$\theta = \arg f'(z_0) = \tan^{-1}\left(\frac{y'}{x'}\right)_{at\,(x_0,y_0)}$

Failure of conformal:- $f'(z)$ has zero of order $(m-1)$ at z_0

$\Rightarrow f'(z_0) = f''(z_0) = \cdots = f^{m-1}(z_0) = 0$ but $f^m(z_0) \neq 0$.

$$f(z) = \sum a_n (z-z_0)^n$$
$$= a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots$$
$$= f(z_0) + 0 + 0 + \cdots + \frac{f^m(z_0)}{m!}(z-z_0)^m$$

$$f(z) - f(z_0) = (z-z_0)^m [a_m + a_{m+1}(z-z_0) + \cdots]$$

$$f(z) - f(z_0) = (z-z_0)^m g(z) \quad \text{and } g(z_0) \neq 0.$$

$$\arg[f(z) - f(z_0)] = m\arg(z-z_0) + \arg(g(z))$$

$$\Rightarrow \arg(\omega - \omega_0) = m\arg(z-z_0) + \arg(g(z))$$

$\lim\limits_{z\to z_0}$ on $\gamma_1 \longrightarrow \beta_1 = m\alpha_1 + \theta_0$

$\lim\limits_{z\to z_0}$ on $\gamma_2 \longrightarrow \beta_2 = m\alpha_2 + \theta_0$

$\boxed{\beta_2 - \beta_1 = m(\alpha_2 - \alpha_1)}$ If $m=1$ then conforma

→ If $f'(z)$ has zero of order $(m-1)$ at z_0 then angle in the image curve magnified by the factor m.

Problem:- If $f(z) = e^{\sin^3 z}$ in z-plane at point $z_0 = \pi/2$
The two curves intersect orthogonality $\alpha = \pi/2$ angle between the curves at the images of z_0 is

(A) $\pi/2$ (B) π (C) $\pi/3$ (D) $2\pi/3$

Solution:- $f'(z) = e^{\sin^3 z} \cdot 3\sin^2 z \cdot \cos z$ $\Rightarrow f'(z)$ has zero of order 1 at $\pi/2$

$f'(\pi/2) = 0, \ f''(\pi/2) \neq 0$

1 increase

$m-1=1$

$\Rightarrow m = 1+1$

The order of $\pi/2$ is 1 $\Rightarrow m = 1+1 = 2$

Then angle $= 2 \cdot \frac{\pi}{2} = \pi = \beta$. $\Rightarrow \boxed{\beta = \pi}$

Problem:- If $f(z) = z^3 \sin^2 z$ at $z_0 = 0$ $\pi/32 = \alpha$, $f^5(0) \neq 0$
image curve angle $\beta = ?$

Solution:- $\boxed{\beta = m\alpha = 5\pi/32}$ $f'(z)$ has zero of order 4 at 0.

Note:- (i) If $\frac{df}{dz} \neq 0$ at that point $\Rightarrow$ Conformal

(ii) If $\frac{df}{dz} = 0$ at that point $\Rightarrow$ not conformal.

Problem:- Check conformal $e^z, e^{z^2}, e^{\sin z}, e^{\cos z}$ at $z=0$.

Solution:- $f(z) = e^z$

$$\frac{df}{dz} = e^z \Rightarrow \left(\frac{df}{dz}\right)_{z_0=0} \neq 0$$

i.e. $e^z \longrightarrow$ conformal

$e^{z^2} \longrightarrow$ not conformal

$e^{\sin z} \longrightarrow$ conformal

$e^{\cos z} \longrightarrow$ not conformal.

Result:- A mapping $w = f(z)$ is conformal at a point z_0 if it is analytic at z_0 and $f'(z_0) \neq 0$.

Problem:- Show that the image of the unit circle $|z|=1$ under the transformation $w = 2z+z^2$ is a cardioid.

Solution:- The transformation can be written as

$$w+1 = (z+1)^2$$

Let $w+1 = \rho e^{i\phi}$, then $|z|=1$ i.e. $z = e^{i\theta}$ is transformed into the curve

$$\rho e^{i\phi} = (e^{i\theta}+1)^2 = [(1+\cos\theta)+i\sin\theta]^2$$

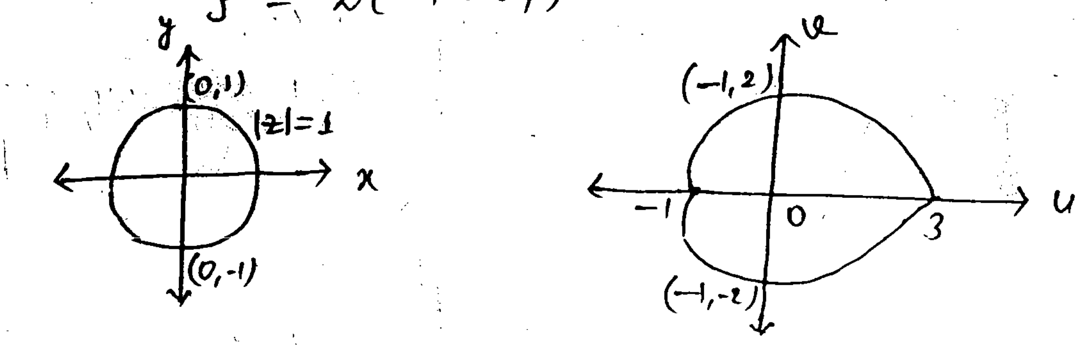
$$\rho e^{i\phi} = 2(1+\cos\theta)e^{i\theta}$$

Hence, $\rho = 2(1+\cos\theta)$, $\phi = \theta$

Eliminating θ, we note that

$\rho = 2(1+\cos\phi)$ is the eqn of the image curve, which is cardioid. The desired cardioid is obtained by shifting the curve along the real axis in the w-plane by one unit to the left. The image curve is

$\rho' = 2(1+\cos\phi)-1$ as shown in figure.



Finally, we comment a little about the transformation

$w = z^n$, $n > 2$. This map is similar to $w = z^2$

Here $w = z^n$, $\rho e^{i\phi} = r^n e^{in\theta}$

This transformation maps the region $r \geq 0$, $\arg z \in [0, \pi/n]$ onto the upper half w-plane.

Result:- Let $f(z)$ be analytic at z_0 s.t. $f'(z_0)=0$. If $f'(z_0)$ has a zero of order $k-1$, $k=1,2,\ldots$ at z_0. Then the mapping $w=f(z)$ magnifies the angle at the vertex z_0 by the factor k.

Proof:- By the given hypothesis, f' has zero of order $k-1$. This means

$$f'(z_0)=f''(z_0)=\cdots=f^{k-1}(z_0)=0$$

Since $f(z)$ is analytic at z_0, it has the representation

$$f(z)=f(z_0)+f'(z_0)(z-z_0)+\frac{f''(z_0)}{2!}(z-z_0)^2+\cdots+\frac{f^{(k-1)}(z_0)}{(k-1)!}(z-z_0)^{k-1}+a_k(z-z_0)^k+\cdots$$

$$f(z)-f(z_0)=a_k(z-z_0)^k+a_{k+1}(z-z_0)^{k+1}+\cdots \qquad a_k\neq 0.$$

$$=(z-z_0)^k\left[a_k+a_{k+1}(z-z_0)+\cdots\right] \qquad a_k\neq 0.$$

$$f(z)-f(z_0)=(z-z_0)^k g(z)$$

where $g(z)$ is analytic at z_0 and $g(z_0)\neq 0$.

Consequently if $w=f(z)$ with $w_0=f(z_0)$, then we obtain

$$\arg(w-w_0)=\arg[f(z)-f(z_0)]$$
$$=k\arg(z-z_0)+\arg g(z) \quad \text{--- (1)}$$

Let 'C' be a smooth arc that passes through z_0.

If $z\to z_0$ along C, then $w\to w_0$ along the image curve Γ and the angles of inclination of tangents to C and Γ are given by the following limits:

$$\theta_0=\lim_{z\to z_0}\arg(z-z_0)$$

and $$\phi_0=\lim_{w\to w_0}\arg(w-w_0)$$

From ①,

$$\phi_0 = k\theta_0 + \delta \;\; ; \;\; \delta = \arg g(z_0) \longrightarrow ②$$

Let C_1 and C_2 be two smooth curves that pass through z_0 and let Γ_1 and Γ_2 be their image curves respectively.
Then from equ^n (2)

$$\boxed{\phi_2 - \phi_1 = k(\theta_2 - \theta_1)}$$

This implies that the angle from Γ_1 to Γ_2 is k-times angle from C_1 to C_2 in magnitude.
However, the orientation is preserved.

Problem:- If $f(z) = z^2 \sin^3 z$ at $z=0$, the angle is $\frac{\pi}{15}$ the at $w_0 = ?$

Solution:- $f(z) = z^2 \sin^3 z$

$$f'(z) = 3z^2 \sin^2 z \cos z + 2z \sin^3 z$$
$$= z \sin^2 z \,(3z\cos z + 2\sin z)$$
$$= z^3 \frac{\sin^2 z}{z^2}\left(3z\cos z + 2z\frac{\sin z}{z}\right)$$
$$f'(z) = z^4\left(\frac{\sin^2 z}{z^2}\right)\left(3\cos z + 2\left(\frac{\sin z}{z}\right)\right)$$

At $z=0$, $f'(z)$ has 0 of order 4 $\quad (k-1 = 4 \Rightarrow k=5)$

By the above result (*)

Angle at $w_0 = 5 \times \frac{\pi}{15} = \frac{\pi}{3}$ Ans

Problem:- If $f(z) = (z-1)^5 \sin z$, at $z=1$ the angle is $\frac{\pi}{12}$, find $w_0 = ?$

Solution:- $f(z) = (z-1)^5 \sin z$

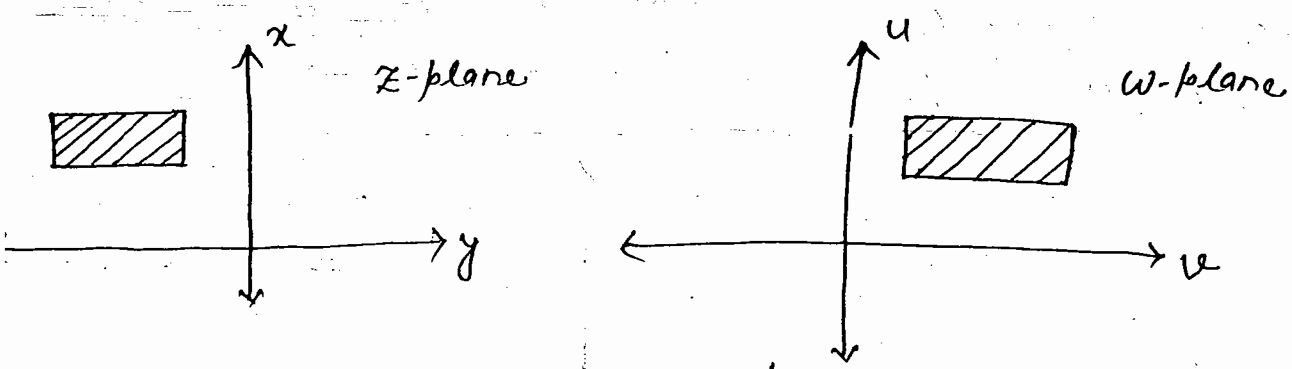
$$f'(z) = (z-1)^5 \cos z + 5(z-1)^4 \sin z$$
$$\Rightarrow f'(z) = (z-1)^4\,[(z-1)\cos z + 5\sin z]$$

$\Rightarrow f'(z)$ has zero of order 4 $\quad [\because k-1 = 4 \Rightarrow k=5]$

Angle at $w_0 = 5 \times \pi \quad = 5\pi$

Some Standard Transformations :-

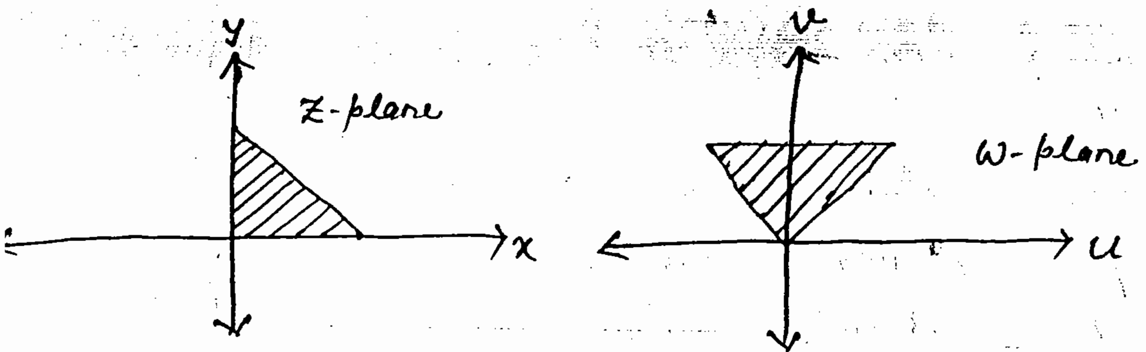
(1). **Translation** :- $w = f(z) = z + a \;;\; a \in \mathbb{C}$



(2) **Rotation** :- $W = f(z) = e^{i\alpha} \cdot z \,,\quad \alpha \in \mathbb{R}$

$\alpha > 0 \longrightarrow$ anticlockwise rotation

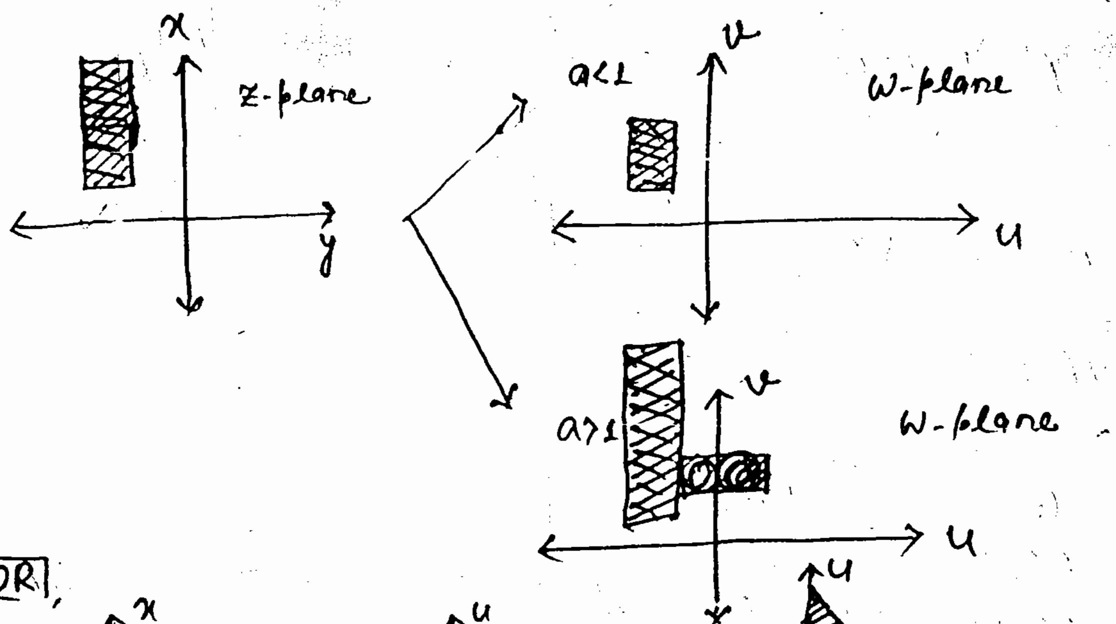
$\alpha < 0 \longrightarrow$ clockwise rotation



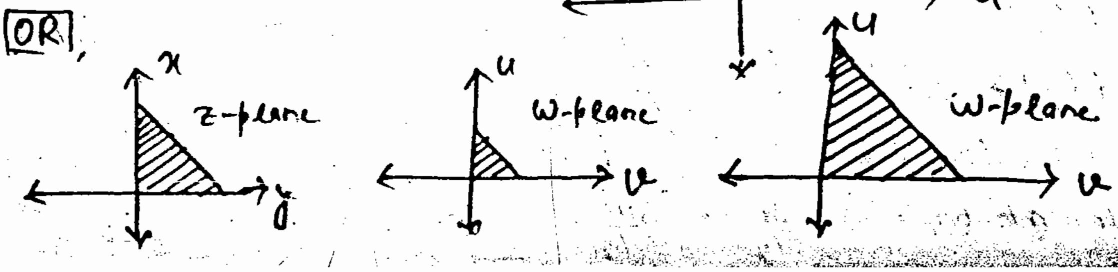
(3) **Magnification or Contraction** :- $W = f(z) = az$, where $a \in \mathbb{R}$.

फैलाव या बढ़ना

संकुचित होना/छोटा होना

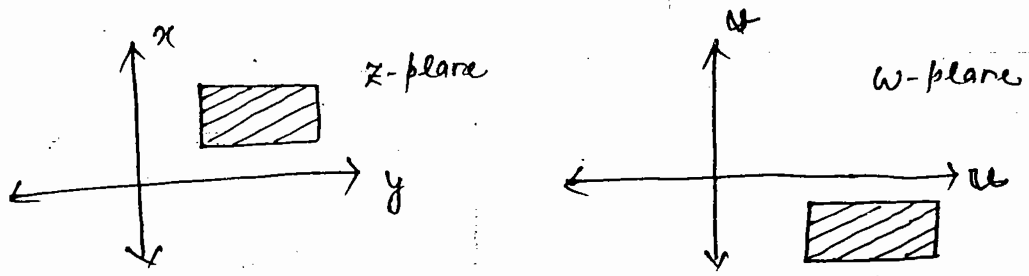


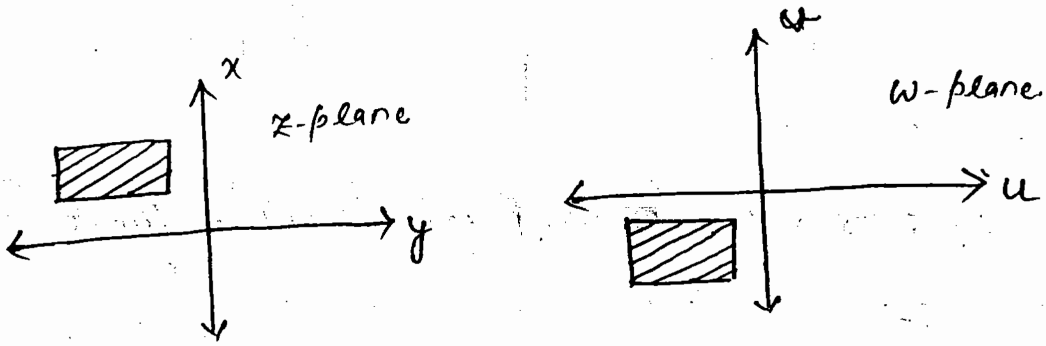
OR,



(4) **Inversion**:- $w = f(z) = \frac{1}{z} = \frac{\bar{z}}{|z|^2}$

It sends to figure from Ist quadrant to IVth quadrant and IInd quadrant to IIIrd quadrant and conversely.





Scale factor of transformation:- Let $w = f(z)$ be a transforma then at any point z_0, $|f'(z_0)|$ is defined as scale factor of $f(z)$ at that point.

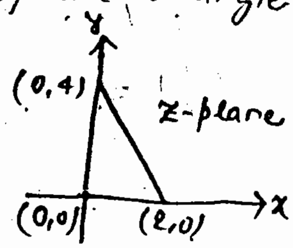
Magnification factor:- The square of scale factor i.e. $|f'(z_0)|$ is called magnification factor.

Problem:- $f(z) = \sqrt{2}\, e^{i\pi/4} z + (2007 + 2008i)$

Find area in z-plane and image of the triangle in w-plan

Solution:- Area in z-plane $= \frac{1}{2} \times 2 \times 4$

$= \underline{\underline{4}}$



Area in w-plane $= |f'(z)|^2 \times 4$

$= (\sqrt{2})^2 \times 4 = \underline{\underline{8}}$

Note:- $f(z) = a e^{i\alpha} . z + b$

$|f'(z)|^2 = a^2$

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$\rightarrow \quad w = e^{z^z} = e^{e^{z\log z}}$

$g\left(\frac{1}{z}\right) = \phi(z) = e^{e^{\frac{1}{z}\log\frac{1}{z}}} = e^{\psi(z)}$

$\lim_{z\to 0} \psi(z)$ does not exist

$\phi(z)$ has essential singularity at $z=0$.

$\Rightarrow g(z)$ " " " $z=\infty$.

$\boxed{\text{Result}}$:- $f(z) = [g(z)]^{f(z)} = e^{f(z)\log g(z)}$ is multivalued and $g(z)=0$ gives branch points.

$\boxed{\text{e.g}}$:- $f(z) = \sin z$ has branch points at $z = n\pi$.

$\boxed{\text{Note}}$ (i) If $f: \mathbb{C} \to \Delta$ (unit disc) is an entire function. Then f is constant by Liouville's theorem.

(ii) When f and f^{-1} is analytic

$\Rightarrow f$ is bianalytic.

(iii) $f^{-1}: \Delta \to \mathbb{C}$ is not possible [$\because f$ becomes constant]

$\boxed{\text{Result}}$:- If $f(z)$ is entire and rate of increase of $|f(z)|$ is slower than some +ve power of $|z|$ then $|f(z)|$ must be polynomial

i.e. $|f(z)| \le M|z|^\alpha \quad \forall z \in \mathbb{C}, \ \alpha \geq 0$

$$\Rightarrow f(z) = \sum_{n=0}^{n=k=[\alpha]} a_n z^n$$

If $0<\alpha<1$ then $f(z)$ is constant.

$\boxed{\text{Proof}}$:- Since $f(z)$ is entire function. Then by Cauchy Integral formula, we have

$$|f^k(0)| = \left| \frac{k!}{2\pi i} \int_{|z|=R} \frac{f(z)}{z^{k+1}}\, dz \right|$$

$$\leq \frac{k!}{2\pi} M R^{\alpha-k-1} \cdot 2\pi R$$

$$\leq k!\, M R^{\alpha-k}$$

$$|f^{k}(0)| = 0 \quad \forall\ k > \alpha \quad \text{as } R \to \infty$$

Since $f(z)$ is entire, we have

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

$$= a_0 + a_1 z + a_2 z^2 + \cdots + a_k z^k + a_{k+1} z^{k+1} + \cdots$$

$$f(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_k z^k + 0 + 0 + \cdots$$

$\Rightarrow$ Polynomial of degree $[\alpha]$.

→ $|f(z)| \leq M |z|^{5/2}$ and $f(z)$ is entire.

$\Rightarrow$ $f(z)$ is a polynomial of degree $\left[\frac{5}{2}\right] = 2$

$$f(z) = a + bz + cz^2$$

$$\& \ f(0) = 0 \Rightarrow a = 0$$

$\Rightarrow$ $f(z) = bz + cz^2$ Ans

→ $|f'(z)| \leq M|z|$

$\Rightarrow$ $f'(z)$ is a linear function.

$$f'(z) = a + bz$$

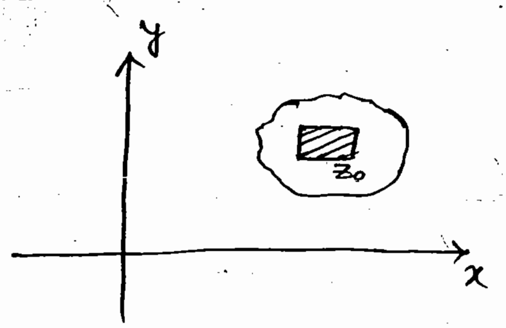
$$f(z) = az + bz^2 + c$$

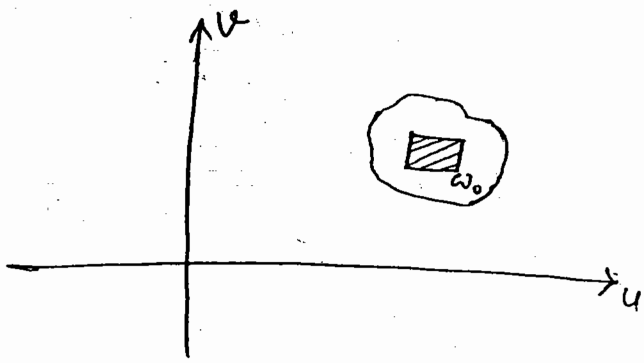
$$f(0) = 0 \Rightarrow c = 0$$

$$f(z) = az + bz^2$$

Residue is zero. Any entire function has zero residue at $z = \infty$.

Jacobian :-





$$\lim_{\Delta A_z \to 0} \frac{\Delta A_w}{\Delta A_z} = J$$

Since $f(z)$ is analytic and conformal in D

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$$

Since $f(z)$ is analytic then $f(z) = u + iv$ satisfies C-R eq^{n}

$$u_x = v_y, \quad u_y = -v_x$$

$$= \begin{vmatrix} u_x & u_y \\ -u_y & u_x \end{vmatrix} = u_x^2 + u_y^2 = |u_x + iu_y|^2$$
$$= |f'(z)|^2 = J$$

→ J may be negative

As $f(z)$ is conformal in D, $f'(z) \neq 0$

⇒ J is non-zero

⇒ $u = u(x,y)$ | $x = x(u,v)$
$v = v(x,y)$ | $y = y(u,v)$

invertible

Note :- (i) This $J = |f'(z)|^2$ is defined as magnification factor of the transformation.

(ii) Whereas the term $|f'(z)|$ is defined as scale factor

→ When $|f'(z)| > 1 \longrightarrow$ Magnification
& $|f'(z)| < 1 \longrightarrow$ Contraction.

$\boxed{\text{Problem}}$:- $f(z) = \sqrt{2}\, e^{i\pi/4} \cdot z + 1 + i$

$$f'(z) = \sqrt{2}\, e^{i\pi/4} = \sqrt{2}\,\frac{(1+i)}{\sqrt{2}} = 1+i$$

$$|f'(z)| = |1+i| = \sqrt{2} \longrightarrow \text{Scale factor.}$$

$$\Rightarrow |f'(z)|^2 = (\sqrt{2})^2 = 2 \longrightarrow \text{Magnification factor.}$$

$\boxed{\text{Result}}$:- If $f(z)$ is conformal at z_0 then $\exists$ a nbd of z_0 in which $f(z)$ is invertible i.e. $\exists$ function $g(z)$ s.t. $f(g(z)) = z$ i.e if $w = f(z)$ then $f(z)$ is bianalytic.

$\boxed{\text{Invariant / Fixed points}}$:- If $w = f(z)$ is analytic in D then $\exists$ complex number $z_0 \in D$ then $f(z_0) = z_0$
i.e zeros of $f(z) - z = 0$
There is only one function $f(z) = z$ in which every comple number is fixed point.

→ Constant function has only one fixed point.
i.e. $f(z) = c$. Here $z = c$ is the only fixed point.

→ Translation has no fixed point.
i.e. $f(z) = z + a$; $a \neq 0$.

→ Polynomial ~~has~~ $P_n = \sum a_n z^n$, $a_n \neq 0$ & $n \geq 1$
then $P(z)$ has exactly n fixed points.

Critical point :- A point $z = z_0$ is said to be critical ~~point of~~ $w = f(z)$ if $f'(z_0) = 0$.

e.g.: $f(z) = e^{2z} - 2iz$, Find critical points.

Now, $f'(z) = 2e^{2z} - 2i = 0 \Rightarrow e^{2z} - i = 0$

$$\Rightarrow e^{2z} = i = e^{i\left(\frac{\pi}{2} + 2n\pi\right)}$$

$$\Rightarrow 2z = i\left(\frac{\pi}{2} + 2n\pi\right)$$

$$\Rightarrow \boxed{z = i\left(\frac{\pi}{4} + n\pi\right)} \; ; n \in N \cup \{0\}$$

If $|z| = 10 \Rightarrow \frac{\pi}{4} + n\pi = 10 \Rightarrow n = \dfrac{10 - \frac{\pi}{4}}{\pi}$

$$n = \frac{10 - \frac{22}{7 \times 4}}{\frac{22}{7}} = \frac{280 - 22}{22 \times 4} = \frac{258}{88} = 2.93$$

i.e. $\boxed{n = 2}$

Bilinear Transformation :- The map $w = \dfrac{az+b}{cz+d}$ —— (1)

$ad - bc \neq 0$, $a, b, c, d \in \mathbb{C}$
is called <u>bilinear transformation</u> or <u>linear fractional transformation</u> or sometimes <u>Möbius transformation</u>
french word

This can also be written as

$$\boxed{w = A + \frac{B}{cz + d}}$$

→ If $ad - bc = 0$ then (1) reduces to constant as

$$w = \frac{a}{c}\left[\frac{z + \cancel{b/a}}{z + \cancel{d/c}}\right] = \frac{a}{c}$$

as $ad - bc = 0 \Rightarrow ad = bc$

$$\Rightarrow \frac{d}{c} = \frac{b}{a}$$

→ A constant function is not linear and hence $ad-bc \neq 0$ is the necessary condition for to be a bilinear transformation

Results:- (i) If $a=1,\ c=0,\ d=1 \Rightarrow$ Translation

(ii) If $|a|=1,\ b=0,\ c=0,\ d=1 \Rightarrow$ Rotation

(iii) If $a \in \mathbb{R}^+,\ b=0,\ c=0,\ d=1 \Rightarrow$ Magnification
$\Rightarrow (|z| \to$ magnification$)$

(iv) If $a=0,\ b=1,\ c=1,\ d=0 \Rightarrow$ Inversion.

Fixed points of B.T.:- Let $w=f(z)=\dfrac{az+b}{cz+d}$; $ad-bc \neq$

Put $\underline{w=z}$ for fixed points,

$$z=\frac{az+b}{cz+d}$$

$$\Rightarrow cz^2+dz-az-b=0$$

$$\Rightarrow cz^2+(d-a)z-b=0$$

Let it has two roots p and q (say),

$$p=\frac{(a-d)+\sqrt{(d-a)^2+4bc}}{2c}$$

$$q=\frac{(a-d)-\sqrt{(d-a)^2+4bc}}{2c}$$

→ If $(a-d)^2+4bc=0$ then B.T. has one fixed point $\dfrac{a-d}{2c}$.

Result:- Every bilinear transformation (except the identity map) has <u>at most</u> two fixed points.

Proof:- If $T(z)=\dfrac{az+b}{cz+d}$; $ad-bc \neq 0$ has a fixed point z,

then $T(z)=z$ or $\dfrac{az+b}{cz+d}=z$

$$\Leftrightarrow cz^2-(a-d)z-b=0$$

Since it is quadratic in z and hence it can have at most two roots.

For the identity map $I(z) = z$, for all points in the domain of definition.

Hence every point of the domain is a fixed point.

This completes the proof.

Canonical or Normal form of the bilinear transformation:-

If B.T. has two distinct fixed points p and q then the B.T. can be expressed as

$$\boxed{\left(\frac{w-p}{w-q}\right) = k\left(\frac{z-p}{z-q}\right) \;;\; k \in \mathbb{C}}$$

Moreover, if B.T. has only one fixed point p i.e. $(a-d)^2+4bc=0$ then the normal form is given by

$$\boxed{\frac{1}{w-p} = k + \frac{1}{z-p} \;;\; k \in \mathbb{C}}$$

Some useful results:-

1). The pair of symmetric points w.r.t. a given curve is mapped into a pair of symmetric points w.r.t. the image curve under a B.T.

2). Under a B.T. circle or line is always mapped into line or circle.

3). A B.T. can have at most two fixed points namely except the identity transformation.

V.I. 4). Identity transformation has infinite fixed points.

V.I. 5). If a B.T. has more than two fixed points then it is identity transformation.

Classification of B.T. on the basis of Normal form :-

(1). **Parabolic** :- A B.T. is said to be parabolic if it has only one fixed point i.e. $(a-d)^2+4bc=0$.

(2) **Elliptic** :- A B.T. with two fixed points i.e $(a-d)^2+4bc\neq 0$ s.t. $|k|=1,\ k\neq 1$ is said to be elliptic i.e. in the normal form k is of the form $k=e^{i\alpha}$; $\alpha\neq 0$

(3) **Hyperbolic** :- If $(a-d)^2+4bc\neq 0$ and $k>0$ i.e k is real, then it is termed as hyperbolic.

(4) **Loxodromic** :- A B.T. that is neither parabolic, elliptic nor hyperbolic is called loxodromic.
i.e it has two fixed points and satisfies the condition

$$k=ae^{i\alpha}\ ;\ \alpha\neq 0,\ a\neq 1.$$

Problem :- Find the fixed points and what is the type of B.T. of the given B.T. $w=\frac{z}{z-2}$

Solution :- $w=\frac{z}{z-2}$,

For fixed pts, Put $w=z$, we have

$$z=\frac{z}{z-2}\Rightarrow z^2-2z-z=0$$

$$\Rightarrow z^2-3z=0\Rightarrow z(z-3)=0$$

$\boxed{z=0,3}$ are the fixed points.

Now, $$\frac{w-0}{w-3}=\frac{\frac{z}{z-2}-0}{\frac{z}{z-2}-3}=\frac{z-0}{-2(z-3)}$$

$$\left(\frac{w-0}{w-3}\right)=\frac{-1}{2}\left(\frac{z-0}{z-3}\right)$$

Here $\boxed{k=-\frac{1}{2}}$

i.e B.T. is loxodromic.

Problem:- Find the fixed points and find the type of B.T. of the given B.T. $w = \dfrac{z}{2-z}$

Solution:- $w = \dfrac{z}{2-z}$

For fixed points, put $w = z$, we get

$$z = \frac{z}{2-z} \Rightarrow z^2 - 2z + z = 0$$

$$\Rightarrow z^2 - z = 0 \Rightarrow z(z-1) = 0$$

$\Rightarrow \boxed{z = 0, 1}$ are the fixed points.

Now,

$$\frac{w-0}{w-1} = \frac{\frac{z}{2-z} - 0}{\frac{z}{2-z} - 1} = \frac{z-0}{2z-2}$$

$$\Rightarrow \frac{w-0}{w-1} = \frac{1}{2}\left(\frac{z-0}{z-1}\right)$$

Here $\boxed{k = \frac{1}{2} > 0}$

Hence B.T. is hyperbolic.

Problem:- Find the fixed points and find the type of B.T. of the given B.T. $w = \dfrac{z+1}{z-1}$

$\dfrac{az+b}{cz+d}$

$a=1$, $b=1$, $c=1$, $d=-1$

Solution:- $w = \dfrac{z+1}{z-1}$

For fixed points, put $w = z$, we get

$$z = \frac{z+1}{z-1} \Rightarrow z^2 - z = z+1 = 0$$

$$\Rightarrow z^2 - 2z - 1 = 0$$

$$\Rightarrow z = \frac{2 \pm \sqrt{4+4}}{2} \Rightarrow \boxed{z = 1 \pm \sqrt{2}}$$ are the fixed points.

$\rightarrow$ $w = \dfrac{az+b}{cz+d}$

$\dfrac{w-p}{w-q} = k\left(\dfrac{z-p}{z-q}\right)$, where p & q are fixed points.

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and $\boxed{k = \dfrac{a-cp}{a-cq}}$ → very imp.

For $p = 1+\sqrt{2}$ & $w = \dfrac{z+1}{z-1}$

$q = 1-\sqrt{2}$

$$k = \frac{1-(1+\sqrt{2})}{1-(1-\sqrt{2})} = \frac{\cancel{1}-\cancel{1}-\sqrt{2}}{\cancel{1}-\cancel{1}+\sqrt{2}} = -1$$

$\Rightarrow \boxed{k=-1}$

Hence B.T. is <u>elliptic</u>.

Problem:- Find the fixed points and find the type of B.T. of the given B.T. $w = \dfrac{(2+i)z-2}{z+i}$

Solution:- $w = \dfrac{(2+i)z-2}{z+i}$

For fixed points, put $w=z$, we get

$$z = \frac{(2+i)z-2}{z+i}$$

$\Rightarrow z^2 + iz = (2+i)z - 2$

$\Rightarrow z^2 + \cancel{iz} = 2z + \cancel{iz} - 2$

$\Rightarrow z^2 - 2z + 2 = 0$

$\Rightarrow z = \dfrac{2 \pm \sqrt{4-8}}{2} \Rightarrow z = \dfrac{2 \pm 2i}{2}$

$\Rightarrow \boxed{z = 1 \pm i}$ are the fixed points.

$$k = \frac{a-cp}{a-cq}$$

$$= \frac{(2+i)-(1+i)}{(2+i)-(1-i)} = \frac{2+\cancel{i}-1-\cancel{i}}{2+i-1+i} = \frac{1}{1+2i} = \frac{1-2i}{5}$$

$\Rightarrow \boxed{k = \dfrac{1-2i}{5}}$

Hence B.T. is <u>loxodromic</u>.

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Problem:- Find the fixed points and find the type of B.T. of the given B.T. $w = \dfrac{2iz+5}{z-2i}$

Solution:- $w = \dfrac{2iz+5}{z-2i}$

For fixed points, put $w=z$, we get

$$z = \frac{2iz+5}{z-2i}$$

$$\Rightarrow z^2 - 2iz = 2iz + 5$$

$$\Rightarrow z^2 - 4iz - 5 = 0$$

$$\Rightarrow z = \frac{4i \pm \sqrt{-16+20}}{2}$$

$\Rightarrow \boxed{z = 2i \pm 1}$ are the fixed points.

Z-plane	W-plane
$\lvert z\rvert \le 1$	$\lvert w\rvert \le 1$
$\lvert z\rvert \le 1$	$\mathrm{Re}\, w \geq 0$ or $\mathrm{Re}\, w \le 0$
$\lvert z\rvert \le 1$	$\mathrm{Im}\, w \geq 0$ or $\mathrm{Im}\, w \le 0$

W-plane	Z-plane
$\lvert w\rvert \le 1$	$\lvert z\rvert \le 1$
$\lvert w\rvert \le 1$	$\mathrm{Re}\, z \le 0$ or $\mathrm{Re}\, z \geq 0$
$\lvert w\rvert \le 1$	$\mathrm{Im}\, z \le 0$ or $\mathrm{Im}\, z \geq 0$

Result:- The set of all bilinear transformation that map the upper half plane in $I(z) > 0$ onto $|w| < 1$ and the boundary $\mathrm{Im}\, z = 0$ onto the boundary $|w| = 1$ is given by

$$\boxed{w = e^{i\alpha} \frac{z - z_0}{z - \bar{z}_0}, \ \mathrm{Im}(z_0) > 0, \ \alpha \in \mathbb{R}}$$

$w = \frac{az+b}{cz+d}$

$w=0 \Rightarrow az+b=0 \Rightarrow z=-b/a$

$w=\infty \Rightarrow cz+d=0 \Rightarrow z=-$

Proof:- Let the bilinear transformation is

$$w = \frac{az+b}{cz+d} \;;\; ad-bc \neq 0$$

Note that the points z and $\bar{z}$ are symmetric (inverse) w.r.t real axis and must correspond to w and $\frac{1}{\bar{w}}$ inverse w.r.t unit circle.

In particular, we observe that $w=0$, $w=\infty$ are the images of the points $z=\frac{-b}{a}$ and $z=\frac{-d}{c}$, respectively and these, being inverse to each other, can be expressed as

$$-\frac{b}{a} = z_0 \quad \text{and} \quad -\frac{d}{c} = \bar{z}_0$$ for some complex constant z_0.

The constant $c \neq 0$, otherwise the points at ∞ will not be mapped on the boundary $|w|=1$. So, can be expressed as

$$\boxed{w = \frac{a}{c}\left(\frac{z+b/a}{z+d/c}\right)} \text{ --- (1)}$$

With this transformation, the bilinear map takes the form

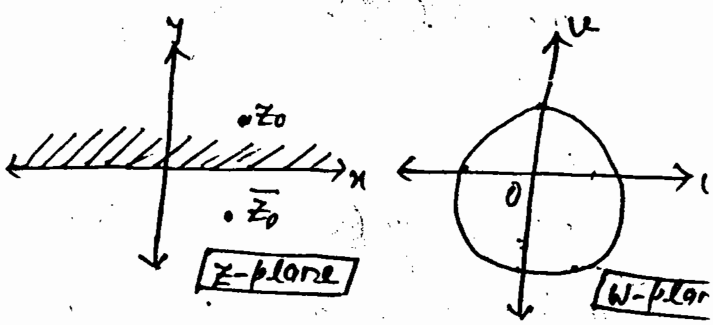
$$w = \frac{a}{c}\left(\frac{z-z_0}{z-\bar{z}_0}\right)$$

Now, $z=0$ on the boundary corresponds to the point on the unit circle $|w|=1$.

So $|w| = 1 = \left|\frac{a}{c}\right| \left|\frac{-z_0}{-\bar{z}_0}\right|$

$\Rightarrow \left|\frac{a}{c}\right| = 1$

$\Rightarrow \boxed{\frac{a}{c} = e^{i\alpha} \;;\; \alpha \in \mathbb{R}}$



Further, the transformation assumes the form,

$$w = e^{i\alpha}\left(\frac{z-z_0}{z-\bar{z}_0}\right) \;;\; \alpha \in \mathbb{R}$$

From (1), the point $z=z_0$ corresponds to $w=0$ and hence z_0 must be in the upper half plane i.e. $\text{Im}(z_0) > 0$.

Hence $\boxed{w = e^{i\alpha}\left(\dfrac{z-z_0}{z-\bar{z}_0}\right) \; ; \; Im(z_0) > 0}$

gives the set of all B.T. which maps $Im(z) \geq 0$ onto $|w| \leq 1$.

Verification :- We can easily verify that the transformation maps upper half plane $Im(z) \geq 0$ onto the unit circular disc $|w| \leq 1$ provided $Im(z_0) > 0$. For this we have

$$w\bar{w} - 1 = e^{i\alpha}\left(\frac{z-z_0}{z-\bar{z}_0}\right) e^{-i\alpha}\left(\frac{\bar{z}-\bar{z}_0}{\bar{z}-z_0}\right) - 1$$

or, $$|w|^2 - 1 = \frac{(z-z_0)(\bar{z}-\bar{z}_0)}{(z-\bar{z}_0)(\bar{z}-z_0)} - 1$$

$$= \frac{(z-\bar{z})(z_0-\bar{z}_0)}{|z-\bar{z}_0|^2}$$

$$= \frac{2i\, Im(z) \cdot 2i\, Im(z_0)}{|z-\bar{z}_0|^2}$$

$$-\frac{4\, Im(z)\, Im(z_0)}{|z-\bar{z}_0|^2}$$

Since $Im(z_0) > 0$, shows that $Im(z) = 0$ is transformed into $|w|^2 - 1 = 0$ i.e. onto $|w| = 1$ and $Im(z) > 0$ is transformed onto $|w|^2 - 1 < 0$ i.e. onto $|w| < 1$.

Hence $Im(z) \geq 0$ is transformed onto $|w| \leq 1$.

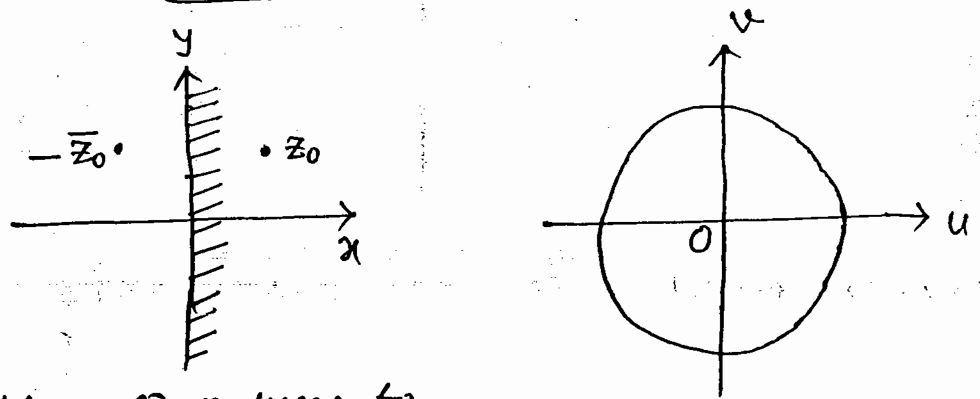
Result :- Find all the bilinear transformations which transform the half plane $Re(z) \geq 0$ onto the unit circular disc $|w| \leq 1$.

Solution :- Let $w = \dfrac{az+b}{cz+d}$ —— (1) ; $ad - bc \neq 0$

be the required B.T. We first note that $c \neq 0$ otherwise the points at ∞ will correspond. The point $w=0$ and $w=\infty$ in the w-plane correspond to the points $z=-b/a$ and $z=-d/c$ in the z-plane. Since $w=0$ and $w=\infty$ are the inverse points w.r.t. unit circle $|w|=1$ therefore $-b/a$ and $-d/c$ must be the inverse points (symmetrical) w.r.t imaginary axis $Re(z)=0$

Thus we may write

$$\boxed{-\frac{b}{a}=z_0\ ,\ -\frac{d}{c}=-\bar{z}_0}$$



Then ① reduces to

$$w=\frac{a}{c}\left(\frac{z-z_0}{z+\bar{z}_0}\right)$$

To find the value of a/c, we use the condition that any point on $Re(z)=0$ (in particular $z=0$) must correspond to a point on $|w|=1$.

$$\therefore\ \left|\frac{a}{c}\right|\left|\frac{0-z_0}{0+\bar{z}_0}\right|=|w|=1$$

$$\Rightarrow\ \left|\frac{a}{c}\right|=1$$

Thus we may take $\frac{a}{c}=e^{i\alpha}$, where α is real.

Consequently the required transformation is

$$\boxed{w=e^{i\alpha}\left(\frac{z-z_0}{z+\bar{z}_0}\right)} \longrightarrow (\text{II})$$

Note that since $z=z_0$ corresponds to $w=0$ which is an interior point of the circle $|w|=1$, the point $z=z_0$ must be a point of the right half plane i.e $Re(z)>0$

with this condition ① is the required transformation.

Verification:- It is easy to see that the transformation ① maps the right half plane $Re(z) \geq 0$ onto the unit circular disc $|W| \leq 1$ provided $Re(z_0) > 0$. For we have

$$W\bar{W} - 1 = e^{i\alpha}\left(\frac{z-z_0}{z+\bar{z}_0}\right) e^{-i\alpha}\left(\frac{\bar{z}-\bar{z}_0}{\bar{z}+z_0}\right) - 1$$

$$\Rightarrow |W|^2 - 1 = \frac{(z-z_0)(\bar{z}-\bar{z}_0)}{(z+\bar{z}_0)(\bar{z}+z_0)} - 1$$

$$= \frac{-(z+\bar{z})(z_0+\bar{z}_0)}{|z+\bar{z}_0|^2}$$

$$\left[\begin{aligned} \because\ & (z+\bar{z}_0)(\bar{z}+z_0) \\ &= (z+\bar{z}_0)(\overline{z+\bar{z}_0}) \\ &= (z+\bar{z}_0)^2 \end{aligned}\right]$$

$$= \frac{-2Re(z)2Re(z_0)}{|z+\bar{z}_0|^2}$$

$$= \frac{-4Re(z)Re(z_0)}{|z+z_0|^2}$$

Since $Re(z_0) > 0$, show that $Re(z) = 0$ is mapped onto $|W|^2 - 1 = 0$ i.e onto $|W| = 1$ and $Re(z) > 0$ is mapped onto $|W|^2 - 1 < 0$ i.e onto $|W| < 1$.

Hence $Re(z) \geq 0$ is transformed onto $|W| \leq 1$.

Result:- The set of all bilinear transformation which map unit disc onto itself is given by

$$\boxed{W = e^{i\alpha}\frac{z-z_0}{1-\bar{z}_0 z}\ ;\ |z_0| < 1,\ a \in \mathbb{R}}$$

such that any point z_0 interior to unit disc is mapped onto the centre of another disc.

Proof:- Clearly $c \neq 0$, otherwise points at ∞ in the two planes would correspond. So, we can be written in the form

$$W = \frac{a}{c}\left(\frac{z+b/a}{z+d/c}\right)$$

Now, $w=0$ and $w=\infty$ are the inverse points of $|w|=1$ and these are the images of $z=-b/a$ and $-d/c$ respectively.

Hence we may write

$$\boxed{-\frac{b}{a}=z_0\,,\quad -\frac{d}{c}=\frac{1}{\bar{z}_0}\,,\quad |z_0|<1}$$

So,

$$w=\frac{a}{c}\left(\frac{z-z_0}{z-\frac{1}{\bar{z}_0}}\right)$$

or,

$$\boxed{w=-\frac{a\bar{z}_0}{c}\left(\frac{z-z_0}{1-\bar{z}_0 z}\right)} \quad \text{——— (1)}$$

The point $z=1$ on the boundary of $|z|=1$ must correspond to a point on the boundary of $|w|=1$.

Thus in the view of (1) and the fact that $|1-z_0|=|1-\bar{z}_0|$ we have

$$1=|w|=\left|\frac{-a\bar{z}_0}{c}\right|\left|\frac{1-z_0}{\bar{z}_0-1}\right|=\left|\frac{a\bar{z}_0}{c}\right|$$

$$\Rightarrow\ -\frac{a\bar{z}_0}{c}=e^{i\alpha}\ ;\ \alpha\in\mathbb{R}$$

This completes the proof.

Verification :-

$$w\bar{w}-1=e^{i\alpha}\left(\frac{z-z_0}{z\bar{z}_0-1}\right)e^{-i\alpha}\left(\frac{\bar{z}-\bar{z}_0}{z_0\bar{z}-1}\right)-1$$

$$=\frac{(z-z_0)(\bar{z}-\bar{z}_0)-(1-\bar{z}_0 z)(1-z_0\bar{z})}{|1-z\bar{z}_0|^2}$$

$$=\frac{(z\bar{z}-1)(1-z_0\bar{z}_0)}{|1-z\bar{z}_0|^2}$$

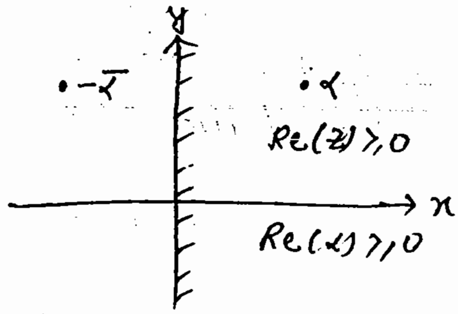
$$=\frac{(|z|^2-1)(1-|z_0|^2)}{|1-z\bar{z}_0|^2}\ ;\ \text{where } |z_i|<1$$

If $|z|=1$ then $|W|=1$. If $|z|<1$, then ~~$|W|^2-1<0$ i.e. $|W|<1$.~~
Hence $|z|\le 1$ is transformed onto $|W|\le 1$ by the transformation (1) provided $|z_0|<1$.

NOTE (1). A B.T. maps pair of symmetrical points into pair of symmetrical points w.r.t corresponding curve.
i.e If $W=\dfrac{az+b}{cz+d}$ maps C in z-plane into γ in w-plane and if z_1 & z_2 are symmetric w.r.t C then $f(z_1)$ and $f(z_2)$ are symmetric w.r.t γ.

NOTE (2) Centre always maps into ∞ in B.T.

Result:- Find the B.T. which transform the $Re(z)\geq 0$ into $|W|\le 1$.



Solution:- Let $W=\dfrac{az+b}{cz+d}$; $c\neq 0$

i.e $W=\dfrac{a}{c}\left(\dfrac{z+b/a}{z+d/c}\right)$ —— (1)

If $z=-b/a \Rightarrow w=0$
If $z=-d/c \Rightarrow W=\infty$
As 0 and ∞ are symmetric w.r.t. $|W|=1$. So $-b/a$ and $-d/c$ must be symmetric w.r.t. $Re(z)=0$ i.e y-axis.
i.e if $\boxed{-b/a=\alpha}$ then $\boxed{-d/c=-\bar{\alpha}}$
Replacing these values in (1), we get

$$W=\frac{a}{c}\left(\frac{z-\alpha}{z+\bar{\alpha}}\right)$$

If $z=0$ is on $Re(z)=0$
$f(0)=W(0)$ is on $|W|=1$

$$W(0)=\frac{a}{c}\left|\frac{0-\alpha}{0+\bar{\alpha}}\right|$$

$$|W(0)|=\left|\frac{a}{c}\right| \Rightarrow \frac{a}{c}=e^{i\lambda}$$

$$\therefore \boxed{W=e^{i\lambda}\left(\frac{z-\alpha}{z+\bar{\alpha}}\right)}$$ for any α and λ —— (1)

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$$w\bar{w}-1 = e^{i\lambda}\left(\frac{z-\alpha}{z+\bar{\alpha}}\right) e^{-i\lambda}\left(\frac{\bar{z}-\bar{\alpha}}{\bar{z}+\alpha}\right) - 1$$

$$= \frac{z\bar{z}-\bar{z}\alpha-z\bar{\alpha}+\alpha\bar{\alpha}}{(z+\bar{\alpha})\,(\overline{z+\bar{\alpha}})} - 1$$

$$= \frac{z\bar{z}-\bar{z}\alpha-z\bar{\alpha}+\alpha\bar{\alpha}-(z+\bar{\alpha})(\bar{z}+\alpha)}{(z+\bar{\alpha})\,(\overline{z+\bar{\alpha}})}$$

$$= \frac{z\bar{z}-\bar{z}\alpha-z\bar{\alpha}+\alpha\bar{\alpha}-(z\bar{z}+\bar{z}\bar{\alpha}+\alpha z+\alpha\bar{\alpha})}{(z+\bar{\alpha})\,(\overline{z+\bar{\alpha}})}$$

$$= \frac{-\bar{z}\alpha-z\bar{\alpha}-\bar{z}\bar{\alpha}-\alpha z}{(z+\bar{\alpha})\,(\overline{z+\bar{\alpha}})}$$

$$= \frac{-\alpha(z+\bar{z})-\bar{\alpha}(z+\bar{z})}{|z+\bar{\alpha}|^2} \qquad [\because z\bar{z}=|z|^2]$$

$$= \frac{-(z+\bar{z})(\alpha+\bar{\alpha})}{|z+\bar{\alpha}|^2}$$

$$= \frac{-2Re(z)\,Re(\alpha)}{|z+\bar{\alpha}|^2}$$

$$w\bar{w}-1 = \frac{-4\,Re(z)\,Re(\alpha)}{|z+\bar{\alpha}|^2}$$

$\Rightarrow |w|^2-1 \le 0 \qquad [\because Re(z)\geq 0,\ Re(\alpha)\geq 0,\ |z+\bar{\alpha}|^2 \geq 0]$

$\Rightarrow \quad |w| \le 1$

Hence $|z|\le 1$ is transformed onto $|w|\le 1$ by the transformation (1) provided $|\alpha|<1$.

→ Similarly we can find B.T. which transform $|z|\le 1$ into $Re(w)\geq 0$.

Cross Ratio:- Let z_1, z_2, z_3, z_4 are fixed complex numbers then their cross ratio taken in this order is given by

$$\boxed{(z_1, z_2, z_3, z_4) = \frac{(z_1-z_2)(z_3-z_4)}{(z_2-z_3)(z_4-z_1)}}$$

$$\frac{4!}{4} = 6$$

Note (i) With given 4 points, we can define at most 6 different cross ratios.

Note (ii) There is <u>unique</u> cross ratio with 4 given points taken in that order.

Result:- The cross ratio is invariant under B.T.

Proof:- Let the B.T. be defined by

$$w = f(z) = \frac{az+b}{cz+d} \; ; \; ad-bc = 1$$

such $w_k = f(z_k)$; $k = 1, 2, 3.$

i.e $w_1 = f(z_1)$, $w_2 = f(z_2)$, $w_3 = f(z_3)$

then we have to show that

$$(w, w_1, w_2, w_3) = (z, z_1, z_2, z_3)$$

Since z_k corresponds to w_k assuming $ad-bc=1$, we have

$$w_k - w = \frac{z_k - z}{(cz_k+d)(cz+d)} \quad \text{using } ad-bc=1.$$

So that any pair of (z, w),

$$\left.\begin{aligned} w_1 - w_2 &= \frac{z_1-z_2}{(cz_1+d)(cz_2+d)} \\ w_2 - w_3 &= \frac{z_2-z_3}{(cz_2+d)(cz_3+d)} \\ w_3 - w &= \frac{z_3-z}{(cz_3+d)(cz+d)} \\ w - w_1 &= \frac{z-z_1}{(cz+d)(cz_1+d)} \end{aligned}\right\} \longrightarrow (1)$$

$$\Rightarrow \frac{2w-i}{i} = \frac{1+3z}{1-z}$$

$$\Rightarrow 2w = \frac{i(1+3z)}{1-z} + i$$

$$\Rightarrow 2w = \frac{i(1+3z+1-z)}{1-z}$$

$$\Rightarrow 2w = \frac{i(2+2z)}{1-z}$$

$$\Rightarrow w = \frac{i(1+z)}{1-z}$$

$$\Rightarrow \boxed{w = -\frac{i(1+z)}{(z-1)}} \text{ Ans}$$

Result:- There exists a unique B.T. which maps three given distinct points z_1, z_2 & z_3 in the extended z-plane onto three specified extended w-plane respectively.

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Schwarz Christoffel Transformation :- This transformation maps a polygon in w-plane onto real axis of z-plane and interior of the polygon onto the upper half of z-plane.

Note :- Open polygon is also considered to be closed.

We are given a polygon in w-plane with vertices $w_1, w_2, \ldots w_n$ and interior angle of these vertices are $\alpha_1, \alpha_2, \ldots \alpha_n$ respectively then the transformation which maps the points $w_1, w_2, \ldots w_n$ onto $x_1, x_2, \ldots x_n$ respectively and interior of the polygon onto upper half of z-plane is given by

$$\boxed{\frac{dw}{dz} = A(z-x_1)^{\frac{\alpha_1}{\pi}-1}(z-x_2)^{\frac{\alpha_2}{\pi}-1} \cdots\cdots (z-x_n)^{\frac{\alpha_n}{\pi}-1}} \quad (1)$$

i.e.
$$\boxed{w = A\int \prod_{i=1}^{n}(z-x_i)^{\frac{\alpha_i}{\pi}-1}\,dz + B} \quad (2)$$

Sum of the exponents is given by

$$\left(\frac{\alpha_1}{\pi}-1\right)+\left(\frac{\alpha_2}{\pi}-1\right)+\cdots\cdots+\left(\frac{\alpha_n}{\pi}-1\right) = \frac{\alpha_1+\alpha_2+\cdots+\alpha_n}{\pi} - n$$

Sum of the interior angle of n-sided polygon $= (n-2)\pi$

$$\Rightarrow \quad \text{Sum of the exponents} = \frac{(n-2)\cancel{\pi}}{\cancel{\pi}} - n = \cancel{n}-2-\cancel{n}$$
$$= -2$$

Note (i) Any three points among $x_1, x_2, \ldots x_n$ can be chosen at will.

(ii) The constants A and B determine the size, orientation and position of the polygon.

(iii) It is convinient to choose one point, say x_n, at ∞ in which case the last factor of (1) & (2) involving x_n is not present.

From ①, we get

$$\frac{(w_1-w_2)(w_3-w)}{(w_2-w_3)(w-w_1)} = \frac{\dfrac{(z_1-z_2)(z_3-z)}{\cancel{(cz_1+d)(cz_2+d)(cz_3+d)(cz+d)}}}{\dfrac{(z_2-z_3)(z-z_1)}{\cancel{(cz_2+d)(cz_3+d)(cz+d)(cz+d_1)}}}$$

$$\Rightarrow \boxed{\frac{(w_1-w_2)(w_3-w)}{(w_2-w_3)(w-w_1)} = \frac{(z_1-z_2)(z_3-z)}{(z_2-z_3)(z-z_1)}}$$

$\Rightarrow$ This is nothing but

$$(w_1, w_2, w_3, w) = (z_1, z_2, z_3, z)$$ Hence proved.

2nd method

Result:- The cross-ratio is invariant under the B.T. and this property can be used in obtaining specific B.T. mapping three points into three other points.

Proof:- Consider the B.T.

$$w = \frac{az+b}{cz+d}$$

If w_k corresponds to z_k, $k = 1, 2, 3$, we have

$$w - w_k = -\left(\frac{az_k+b}{cz_k+d}\right) + \left(\frac{az+b}{cz+d}\right)$$

$$\Rightarrow w - w_k = \frac{(z-z_k)(ad-bc)}{(cz_k+d)(cz+d)}$$

Then $$w - w_1 = \frac{(ad-bc)(z-z_1)}{(cz_1+d)(cz+d)}$$

$$w_3 - w = \frac{(ad-bc)(z_3-z)}{(cz_3+d)(cz+d)}$$

Replacing w by w_2 and z by z_2, we have

$$w_1 - w_2 = \frac{(ad-bc)(z_1-z_2)}{(cz_2+d)(cz_1+d)}$$

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$$W_2 - W_3 = \frac{(ad-bc)(z_2-z_3)}{(cz_2+d)(cz_3+d)}$$

By dividing, we get:

$$\boxed{\frac{(W_1-W_2)(W_3-W)}{(W_2-W_3)(W-W_1)} = \frac{(z_1-z_2)(z_3-z)}{(z_2-z_3)(z-z_1)}}$$

Problem:- Find a B.T. which maps $z=0, -i, -1$ into $w = i, 1, 0$ respectively.

Solution:- $\frac{(W_1-W_2)(W_3-W)}{(W_2-W_3)(W-W_1)} = \frac{(z_1-z_2)(z_3-z)}{(z_2-z_3)(z-z_1)}$

Put $W_1 = i$, $W_2 = 1$, $W_3 = 0$

& $z_1 = 0$, $z_2 = -i$, $z_3 = -1$

Then cross ratio becomes

$$\frac{(i-1)(0-w)}{(1-0)(w-i)} = \frac{(0+i)(-1-z)}{(-i+1)(z-0)}$$

$$\Rightarrow \cancel{-}\frac{(i-1)w}{(w-i)} = \frac{\cancel{-}(1+z)i}{(1-i)z}$$

$$\Rightarrow \frac{w}{w-i} = \frac{(1+z)i}{-(1-i)^2 z}$$

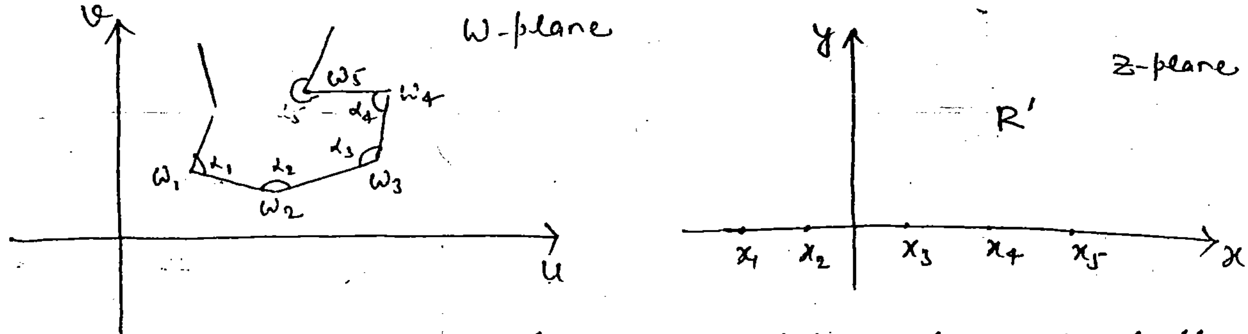
$$\Rightarrow \frac{w}{w-i} = \frac{(1+z)i}{-z(\cancel{1}-\cancel{1}-2i)}$$

$$\Rightarrow \frac{w}{w-i} = \frac{\cancel{i}(1+z)}{2\cancel{i}z}$$

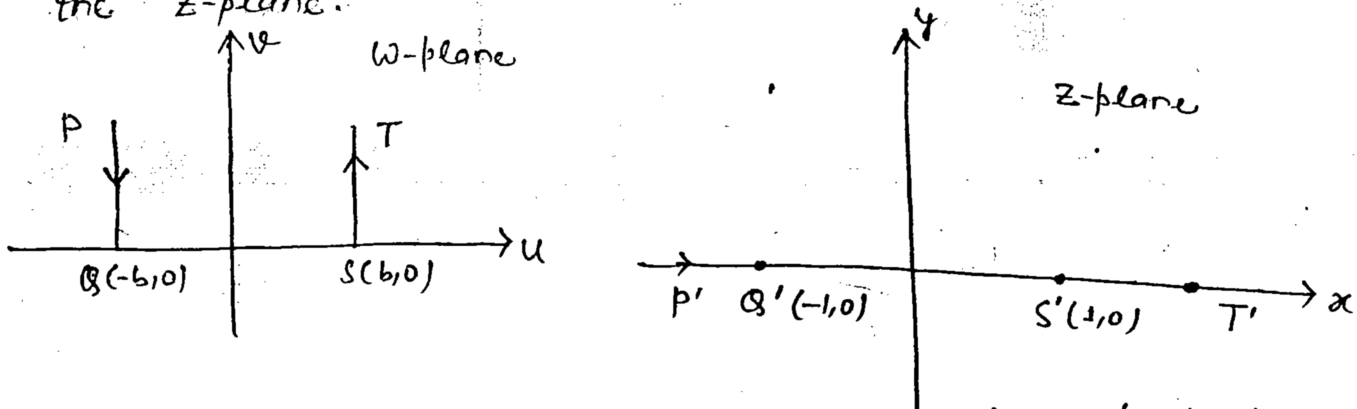
$$\Rightarrow \frac{w}{w-i} = \frac{1+z}{2z}$$

$$\Rightarrow \frac{w+w-i}{i} = \frac{1+z+2z}{1+z-2z}$$ [using componendo and dividendo]

(iv) Infinite open polygons can be considered as limiting case of closed polygons.



Problem:- Determine a function which maps each of the indicated region in the w-plane onto the upper half of the z-plane.



Let points P, Q, S, and T maps respectively into P', Q', S' and T'. We can consider PQST as a limiting case of a polygon (Δ) with two vertices at Q and S and the third vertex P or T at ∞.

By the Schwarz - Christoffel transformation since the angles at Q and S are equal to $\pi/2$, we have

$$\frac{dw}{dz} = A(z+1)^{\frac{\pi/2}{\pi}-1}(z-1)^{\frac{\pi/2}{\pi}-1}$$

$$\Rightarrow \frac{dw}{dz} = \frac{A}{\sqrt{z^2-1}} = \frac{A'}{\sqrt{1-z^2}} \qquad \text{where } A' = iA$$

Integrating, we have

$$w = A'\int \frac{dz}{\sqrt{1-z^2}} + B$$

$$w = A'\sin^{-1} z + B$$

When $z=1$, $w=b$.

Hence $b = A'\sin^{-1}(1) + B$

$$\Rightarrow b = A'\frac{\pi}{2} + B \quad \text{——— (1)}$$

When $z=-1$, $w=-b$

Hence $-b = A'\sin^{-1}(-1) + B$

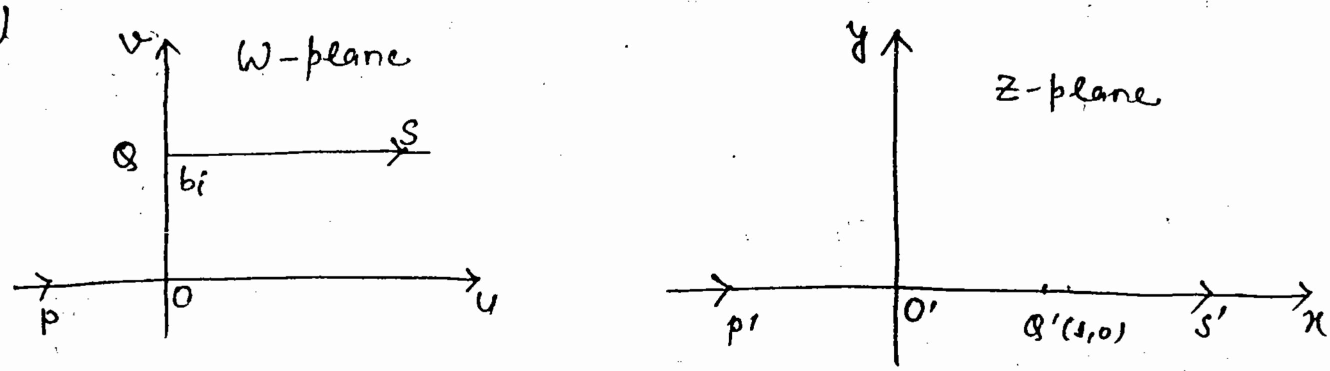
$$\Rightarrow -b = -A'\frac{\pi}{2} + B \quad \text{——— (2)}$$

Solving (1) and (2) simultaneously, we find

$B=0$, $A' = \frac{2b}{\pi}$, Then

$$w = \frac{2b}{\pi}\sin^{-1}z \quad \text{or} \quad \boxed{z = \sin\frac{\pi w}{2b}}$$

(ii)



Let the points P, O, Q $[w=bi]$ and S map into $P', O', Q'(z=1)$ and S' respectively. Note that P, S, P', S' are at ∞ while O and O' are the origin $[w=0$ and $z=0]$ of the w and z-planes. Since the interior angles at O and Q are $\frac{\pi}{2}$ and $\frac{3\pi}{2}$ respectively. We have by Schwarz-Christoffel transformation

$$\frac{dw}{dz} = A(z-0)^{\frac{\pi/2}{\pi}-1}(z-1)^{\frac{3\pi/2}{\pi}-1}$$

$$\Rightarrow \frac{dw}{dz} = A\sqrt{\frac{z-1}{z}}$$

$$\Rightarrow \frac{dw}{dz} = k\sqrt{\frac{1-z}{z}} \qquad \text{where } k = iA$$

Then $w = k \int \sqrt{\frac{1-z}{z}}\, dz$

To integrate this, let $z = \sin^2\theta$ and obtain

$$w = 2k \int \cos^2\theta \, d\theta$$

$$\Rightarrow w = k \int (1+\cos 2\theta)\, d\theta$$

$$\Rightarrow w = k\left(\theta + \frac{1}{2}\sin 2\theta\right) + B$$

When $z=0$, $w=0$, so that $B=0$

When $z=1$, $w=bi$ so that $bi = \frac{k\pi}{2}$ or $k = \frac{2bi}{\pi}$

Then the required transformation is

$$\boxed{w = \frac{2bi}{\pi}\left(\sin^{-1}\sqrt{z} + \sqrt{z(1-z)}\right)}$$

Transformation of boundaries in parametric form:-

Let C be a curve in the z-plane with parametric equation $x = F(t)$, $y = G(t)$.

Show that the transformation

$$z = F(w) + iG(w)$$ maps curve C onto the real axis of the w-plane.

If $z = x+iy$, $w = u+iv$, the transformation can be written

$$x+iy = F(u+iv) + iG(u+iv)$$

Then $v=0$ [the real axis of the w-plane] corresponds to

$x+iy = F(u) + iG(u)$

i.e. $\boxed{x = F(u),\quad y = G(u)}$ which represents the curve C.

Problem:- Find a transformation which maps the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ in the z-plane on the real axis of the w-plane.

Solution:- A set of parametric equations for the ellipse is given by $x = a\cos t$, $y = b\sin t$, where $a>0$, $b>0$. Then the required transformation is

$\boxed{z = a\cos w + ib\sin w}$

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UNIT-5 POWER SERIES

A series having the form

$$a_0 + a_1(z-a) + a_2(z-a)^2 + \cdots\cdots = \sum_{n=0}^{\infty} a_n(z-a)^n \quad \text{---}(1)$$

is called a power series in $(z-a)$.
We shall sometimes indicate (1) briefly by $\sum a_n(z-a)^n$.

Clearly the power series (1) converges for $z=a$, and this may indeed be the only point for which it converges.
In general, however the series converges for other points as well. In such a case we can show that there exists a positive number R such that (1) converges for $|z-a|<R$ and diverges for $|z-a|>R$, while for $|z-a|=R$ it may or may not converge.

Geometrically if Γ is a circle of radius R with centre at $z=a$, then the series (1) cgs at all points inside Γ and dgs at all points outside Γ, while it may or may not converge on the circle Γ.
We can consider the special cases $R=0$ and $R=\infty$ respective to be the cases where (1) converges only at $z=a$ or converges for all values of z. Because of this geometric interpretation, R is often called the radius of convergence of (1) & the corresponding circle is called the circle of convergence.

Power Series :- $\sum_{n=0}^{\infty} a_n z^n$ or $\sum_{n=0}^{\infty} a_n (z-a)^n$ is called power series.

Property :-

1). If converges uniformly to $f(z)$ then $f(z)$ is analytic in some nbd of 0 or atleast differentiable at $z=0$.

2). If $f(z)$ be the sum of series $\sum a_n z^n$ and the power series converges uniformly to $f(z)$ then the distence of the nearest singularity of $f(z)$ from $z=0$ is defined as radius of convergence (R.O.C) if it is R then $|z| \le R$ is said to be domain of convergence.

Interview

3). Taylor series can not have R.O.C. zero. But Power series can have R.O.C. zero.

Interview — What is basic differnce between Taylor's series & Power series.

When we expand a function as a Taylor series, we expand in a nbd of regular point.

Radius of Convergence (R.O.C) :- If $\exists R > 0$ s.t. for every $z \in |z-a| < R$ of $\sum a_n (z-a)^n$ converges uniformly and may fail to be cgt on $|z-a| = R$ but for any $\delta > 0$ $\exists z_0 \in |z-a| < R+\delta$ s.t. $\sum a_n (z-a)^n$ fails to be cgt at z_0 then R is said to be ROC of $\sum a_n (z-a)^n$ & the region $|z-a| < R$ is called region of convergence.

We define

(i) $$\frac{1}{R} = \limsup_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|$$

(ii) $$\frac{1}{R} = \limsup_{n \to \infty} |a_n|^{1/n} \quad \text{——} (A)$$

(A) is known as Hadamard formula. The quantity R is called the ROC of the power series.

Infact the ROC of power series $\sum a_n(z-a)^n$ is the reciproc of limit superior of sequence b_n.

Where $b_n = \left|\frac{a_{n+1}}{a_n}\right|$ or $|a_n|^{1/n}$ which exist.

Problem:- Find the R.O.C. of the series.

(i) $\sum_{n=0}^{\infty} n z^n \rightarrow \boxed{ROC = 1}$

(ii) $\sum_{n=0}^{\infty} n^n z^n \rightarrow \boxed{ROC = 0}$

(iii) $\sum_{n=0}^{\infty} 2^n z^n \rightarrow \boxed{ROC = 1/2}$

(iv) $\sum_{n=0}^{\infty} \frac{z^n}{n!} \rightarrow \boxed{ROC = \infty}$

(v) $\sum_{n=0}^{\infty} \frac{z^n}{n} \rightarrow \boxed{ROC = 1}$

(vi) $\sum_{n=0}^{\infty} 2^n z^{2n} \rightarrow \boxed{ROC = 1/\sqrt{2}}$

(vii) $\sum_{n=0}^{\infty} \frac{z^{2n}}{2^n} \rightarrow \boxed{ROC = \sqrt{2}}$

(viii) $\sum_{n=0}^{\infty} z^{n!} \rightarrow \boxed{ROC = 1}$.

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(ix) $\sum_{n=0}^{\infty} 2^n z^{n!} \rightarrow \boxed{ROC = 1/2}$

(x) $\sum_{n=0}^{\infty} z^n \rightarrow \boxed{ROC = 1}$

(xi) $\sum_{n=0}^{n} a_n z^n$

Where $a_n = \begin{cases} (1+2i)^n ; & n \text{ is odd} \\ (2+3i)^n ; & n \text{ is even} \end{cases}$

$b_n = |a_n|^{1/n} = \begin{cases} \sqrt{5} ; & n \text{ is odd} \end{cases} \rightarrow \boxed{ROC = 1/\sqrt{13}}$

(xii) $\sum_{n=0}^{\infty} \frac{1}{2^n} z^n \rightarrow \boxed{ROC = 2}$

(xiii) $\sum_{n=0}^{\infty} a_n z^n$

where $a_n = \begin{cases} 5^n ; & \text{when } n \text{ is even} \\ 3^n ; & \text{when } n \text{ is odd} \end{cases}$

$\downarrow \boxed{ROC = 1/5}$

Solution (i) $\sum_{n=0}^{\infty} n z^n$

$$\frac{1}{R} = \limsup_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \limsup_{n\to\infty}\left|\frac{n+1}{n}\right| = \limsup_{n\to\infty}\left|1+\frac{1}{n}\right| = 1$$

$\therefore \boxed{R = 1}$ Ans

(ii) $\sum_{n=0}^{\infty} n^n z^n$

$$\frac{1}{R} = \limsup_{n\to\infty}|a_n|^{1/n} = \limsup_{n\to\infty}|n^n|^{1/n} = \limsup_{n\to\infty}|n| = \infty$$

$\therefore \boxed{R=0}$

(iii) $\sum_{n=0}^{\infty} 2^n z^n$

$$\frac{1}{R} = \limsup_{n\to\infty}\left|\frac{2^{n+1}}{2^n}\right| = \limsup_{n\to\infty}|2| = 2$$

$\therefore \boxed{R = 1/2}$

$a_n = 2^n$

$\frac{1}{R} = \lim_{n\to\infty} |a_n|^{1/n}$

$\Rightarrow \frac{1}{R} = \lim_{n\to\infty} (2^n)^{1/n}$

$\Rightarrow \frac{1}{R} = \lim_{n\to\infty} (2)$

$\Rightarrow R = 1/2$

(iv) $\sum_{n=0}^{\infty} \frac{z^n}{n!}$

$$\frac{1}{R} = \limsup_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \limsup_{n\to\infty}\left|\frac{1}{n+1}\right| = 0$$

$\therefore \boxed{R=\infty}$

(v) $\sum_{n=0}^{\infty} \frac{z^n}{n}$

$$\frac{1}{R} = \limsup_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \limsup_{n\to\infty}\left|\frac{n+1}{n}\right| = \limsup_{n\to\infty}\left|1+1/n\right|$$

$= 1$

$\therefore \boxed{R=1}$

(vi) $\sum_{n=0}^{\infty} 2^n z^{2n}$

Put $z^2 = t$, we get the series as

$\sum 2^n t^n$

$\therefore \frac{1}{R'} = \lim_{n\to\infty} \sup \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \sup \left|\frac{2^{n+1}}{2^n}\right| = 2$

$\Rightarrow R' = 1/2$

$\therefore |t| < 1/2$

$\Rightarrow |z^2| < 1/2$

$\Rightarrow |z| < 1/\sqrt{2}$.

$\Rightarrow$ ROC of $\sum 2^n z^{2n}$ is $\frac{1}{\sqrt{2}}$

$\therefore \boxed{R = \frac{1}{\sqrt{2}}}$

$\sum_{n=0}^{\infty} (\sqrt{2})^{2n} z^{2n}$

$\frac{1}{R} = \sqrt{2}$

$\boxed{R = \frac{1}{\sqrt{2}}}$

(vii) $\sum_{n=0}^{\infty} \frac{z^{2n}}{2^n}$

$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ does not exist. We cannot apply the formula

$\frac{1}{R} = \lim_{n\to\infty} \sup \left|\frac{a_{n+1}}{a_n}\right|$

For this we proceed through the def^n.

Note that

$$a_n = \begin{cases} 0 & ; n = 2k-1 \\ 2^{-n} & ; n = 2k \end{cases} \qquad k = 1, 2, 3, \ldots$$

$\sum_{n=0}^{\infty} \frac{z^{2n}}{(\sqrt{2})^{2n}}$

$\frac{1}{R} = \frac{1}{\sqrt{2}}$

$\boxed{R = \sqrt{2}}$

NOW, $\lim_{n\to\infty} \sup |a_n|^{1/n} = \lim_{k\to\infty} |2^{-2k}|^{1/2k} = \frac{1}{2}$

& $\lim_{n\to\infty} \inf |a_n|^{1/n} = \lim_{k\to\infty} |a_{2k-1}|^{1/2k-1} = 0$

Hence $|z|^2 = 2 \Rightarrow \boxed{R = \sqrt{2}}$

Note:- The procedure is simplified if z^2 is replaced by s.
Then $|z^2| = |s| = R$.

(viii) $\sum_{n=0}^{\infty} z^{n!} = z + z^2 + z^6 + z^{24} + \cdots\cdots$

missing term
$z^3 + z^4 + z^5 + z^7 + \cdots$
0

$\therefore \quad a_n = \begin{cases} 1 & ; \text{ if } \exists\, k \in \mathbb{N} \text{ s.t. } n = k! \\ 0 & ; \text{ otherwise} \end{cases}$

$\frac{1}{R} = \limsup_{n\to\infty} a_n = 1$

$\Rightarrow \boxed{R = 1}$

$\sum_{n=0}^{\infty} 1^{n!} z^{n!}$

$\frac{1}{R} = 1$ $\quad \boxed{R=1}$

(ix) $\sum_{n=0}^{\infty} 2^n . z^{n!} = 2^0 . z^{0!} + 2^1 z^{1!} + 2^2 z^{2!} + 2^3 z^{3!} + 2^4 z^{4!} + \cdots$
$= \cancel{z} + 2! z + 4 z^2 + 8 z^6 + 16 z^{24} + \cdots$

Here $a_n = \begin{cases} 2^k & ; n = k! \\ 0 & ; \text{ otherwise} \end{cases}$

$(a_n)^{1/n} = \begin{cases} 2^{k/n} & ; n = k! \\ 0 & ; \text{ otherwise} \end{cases}$

$\frac{1}{R} = \limsup_{n\to\infty} (2^{k/n}) = 2$

$\therefore \boxed{R = \frac{1}{2}}$

(x) $\sum_{n=0}^{\infty} z^n$

$\frac{1}{R} = \limsup_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = 1$

$\therefore \boxed{R = 1}$

(xi) $\sum_{n=0}^{\infty} a_n z^n$

where $a_n = \begin{cases} (1+2i)^n & ; n \text{ is odd} \\ (2+3i)^n & ; n \text{ is even} \end{cases}$

$$b_n = |a_n|^{1/n} = \begin{cases} \sqrt{5} & ; n \text{ is odd} \\ \sqrt{13} & ; n \text{ is even} \end{cases}$$

$$\lim_{n\to\infty} \sup b_n = \sqrt{13}$$

$$\Rightarrow \boxed{ROC = \frac{1}{\sqrt{13}}}$$

(xii) $\sum_{n=0}^{\infty} a_n z^n$

where $a_n = 1/2^n$.

$$\frac{1}{R} = \lim_{n\to\infty} \sup |a_n|^{1/n} = \lim_{n\to\infty} \sup 1/2 = \lim_{n\to\infty} \frac{1}{2} = \frac{1}{2}$$

$$\Rightarrow \boxed{R = 2}$$

(xiii) $\sum_{n=0}^{\infty} a_n z^n$

where $a_n = \begin{cases} 5^n & ; \text{when } n \text{ is even} \\ 3^n & ; \text{when } n \text{ is odd} \end{cases}$

$$\frac{1}{R} = \lim_{n\to\infty} \sup |a_n|^{1/n} = \lim_{n\to\infty} \sup \{5, 3\} = 5$$

i.e. $\boxed{R = 1/5}$

→ Let $z=a$ be a regular point of $f(z)$ then we have Taylor's expansion in the form $f(z) = \sum a_n (z-a)^n$ then R.H.S. is a power series which converges uniformly to $f(z)$ at every point of some nbd. of a

i.e. $\forall z \in |z-a| < R$

and fails to be cgt at least one point for any increment in R.

$\boxed{\text{Note}}$ ~~(i) The series may fail to be cgt on some point of boundary.~~

(ii) If power series converges at every point of $D: |z-a|<R$ & R is ROC then there must exist at least one point on the boundary where power series fails to be cgt.

(iii) If $f(z)$ is expanded in the nbd of $z=a$ where 'a' is the regular point then ROC of the expansion is the distance of the nearest singularity of $f(z)$ from a.

$\boxed{\text{e.g}}$:- $f(z) = \sin\frac{1}{z}$; $z = \pi/2$

$\frac{\pi}{2}$ is ROC $\quad \because \sin\frac{1}{z} = \sum a_n (z-\pi/2)^n$

singularity at $z=0$.

$\boxed{\text{Note}}$:- $\sum_{n=0}^{\infty} a_n z^n$ has R.O.C is R then

(i) $\sum a_n^2 z^n$ has R.O.C is R^2

Imp (ii) $\sum a_n z^{kn}$ has R.O.C. is $R^{1/k}$

(iii) $\sum \frac{a_n}{n!} z^n$ has R.O.C. is ∞

(iv) $\sum (1+z_0)^n a_n z^n$ has R.O.C. is $R/(1+z_0)$

(V) $\sum k^n a_n z^n$ has R.O.C. is R/k

Result :- **Cauchy Hadamard Theorem** :- For every power series $\sum_{n=0}^{\infty} a_n z^n$ there exists a number R s.t. $0 \le R \le \infty$ i.e. $R \in [0, \infty$ with the following properties :-

(i) The series converges absolutely for every z s.t. $|z| < R$.

(ii) The series diverges if $|z| > R$.

V.V.I.

Result :- The power series $\sum_{n=0}^{\infty} n a_n z^{n-1}$, obtained by differenti the power series $\sum_{n=0}^{\infty} a_n z^n$, has the same ROC as the original series $\sum_{n=0}^{\infty} a_n z^n$.

Result :- The sum function $f(z)$ of the power series $\sum a_n z^n$ represents an analytic function inside the circle of convergence.

Corollary :- The sum of a power series is continuous in any region which lies entirely within the circle of convergence.

e.g. :- $f(z) = 1 + 2z + 3z^2 + 4z^3 + \cdots$

$$= (1-z)^{-2}$$

$$= \frac{1}{(1-z)^2}$$

Singularity at $z = 1$

$$|z| < 1$$

$$\Rightarrow \boxed{ROC = 1}$$

$$\int f(z)\,dz = z + z^2 + z^3 + \cdots$$

$$= \frac{z}{1-z}$$

singularity at $z = 1$

[Problem]:- $f(z) = \sum a_n z^n$ has R.O.C is R

Remember

Define $g(z) = \sum \frac{a_n}{n!} z^n$ s.t. $g(z)$ is entire

then $g(z)$ has ROC is ∞

[Calculation]:- $a_{n+1}^2 \cdot a_n = a_{n+2}$, find $a_n = ?$

$$2\log(a_{n+1}) + \log(a_n) = \log(a_{n+2})$$

Put $\log(a_n) = b_n$

$$\Rightarrow 2b_{n+1} + b_n = b_{n+2}$$

$$\Rightarrow b_{n+2} - 2b_{n+1} - b_n = 0$$

$$(E^2 - 2E - 1) b_n = 0$$

$$E = \alpha, \beta$$

then $b_n = c_1 e^{\alpha} + c_2 e^{\beta}$

$$a_n = e^{c_1 e^{\alpha} + c_2 e^{\beta}}$$

$$\Rightarrow \boxed{a_n = e^{c_1 e^{\alpha}} \cdot e^{c_2 e^{\beta}}}$$

V.Imp

[Problem]:- Let $\sum a_n z^n$ has co-efficient with recurrence relation given by $a_0 = 1$, $a_1 = -1$

and $3a_n + 4a_{n-1} - a_{n-2} = 0$

Find the ROC of the power series.

[Solution]:- $3a_n + 4a_{n-1} - a_{n-2} = 0$

$$\Rightarrow 3 + 4s - s^2 = 0$$

Let $\phi(z) = 3 + 4z - z^2$

$$f(z) = \sum a_n z^n$$

$$\phi(z)\cdot f(z) = (3+4z-z^2)(a_0+a_1z+a_2z^2+a_3z^3+\cdots\cdots)$$

$$= 3a_0 + (3a_1+4a_0)z + (3a_2+4a_1-a_0)z^2 + \cdots$$

$$= 3a_0 + (3a_1+4a_0)z + \sum(3a_n+4a_{n-1}-a_{n-2})z^n$$

Given $a_0 = 1$, $a_1 = -1$

$$= 3 + z + \sum 0\cdot z^n$$

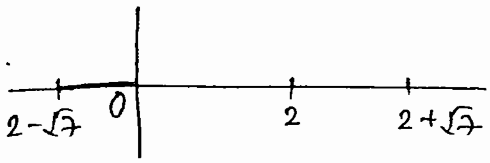
$$\phi(z)\cdot f(z) = 3+z$$

$$\Rightarrow f(z) = \frac{3+z}{\phi(z)}$$

$$\boxed{f(z) = \frac{3+z}{3+4z-z^2}} \text{ Ans}$$

$$z^2-4z-3=0$$

$$\Rightarrow z = 2 \pm \sqrt{7}$$



$2-\sqrt{7}$ is nearest to 0

then $\boxed{\text{R.O.C is } 2-\sqrt{7}}$ Ans

V.Imp

Problem:- Discuss the R.O.C. of a power series whose nth co-efficient of $\sum a_n z^n$ is $a_{n+2} = a_{n+1} - a_n$

$a_0 = a_1 = 1$.

$\boxed{\text{R.O.C} = \frac{-1 \pm \sqrt{5}}{2}}$

V.Imp

Problem:- Let $f(z) = 1 + 2z + 3z^2 + 4z^3 + \cdots\cdots$

(or), $f(z) = \sum n z^{n-1}$ for $|z| < 1$

define a sequence of real numbers $a_0, a_1, a_2, \cdots$

s.t. $f(z) = \sum a_n (z+2)^n$

Now, find the R.O.C. of $\sum a_n z^n$

Hint:. $\sum z^n = \sum n z^{n-1}$

Result:- If $f(z)=\sum_{n=0}^{\infty} a_n z^n$ converges in $|z|<R$, then the derivatives of all orders exist in $|z|<R$ and

$$\boxed{f^{k}(z)=\sum_{n=k}^{\infty} n(n-1)(n-2)\text{----}(n-k+1)a_n z^{n-k}\;;\; k=0,1,2,\text{---}}$$

→ Every power series represents an analytic function interior to its circle of convergence, and it can be integrated and differentiated term by term. Moreover, the derivatives of all orders of a power series also give rise to analytic functions interior to the same circle of convergence.

A complex function is even if $f(-z)=f(z)$ in D and odd if $f(-z)=-f(z)$ in D.
Note that $f(z)=z^2$ is even in $|z|\leq 2$ but we cannot say anything about $f(z)=z^2$; $z=2e^{i\theta}$; $0\leq\theta\leq\pi$.
The domain must be symmetrical such that z and $-z$ must be in D.

Remark:- Let $f(z)=\sum_{n=0}^{\infty} a_n z^n$. Then

(i) $a_n=0$ for all $n=0,2,4,6,\text{----}$,
if $f(z)$ is an odd function

(ii) $a_n=0$ for all $n=1,3,5,7,\text{-----}$,
if $f(z)$ is an even function.

→ Note that a power series can behave in any of the following three ways on its circle of convergence Γ:.

(i) The series may diverge everywhere on Γ, as the geometric series $\sum_{n=0}^{\infty} z^n$.

(ii) The series may converge at some of the points of Γ and diverge at other points of Γ, as the logarithmic series $\sum_{n=1}^{\infty} (-1)^n \frac{z^n}{n}$, which cgs for $z=1$ & dgs for $z=-1$.

(iii) The series may converge for all points on Γ, as the series $\sum_{n=0}^{\infty} z^2/n^2$ which converges for all $|z|=1$.

The convergence of a power series at a point $\delta \in \Gamma$ implies in terms of the behaviour of the sum of the series in a nbd of δ.

$\boxed{\text{Abel's Limit Theorem}}$:- If $\sum_{n=0}^{\infty} a_n$ converges, then the power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ with $R=1$ tends to $f(1)$ as $z \to 1$, provided $\frac{|1-z|}{1-|z|}$ remains bounded.

$\boxed{\text{Schwarz Lemma}}$:- Suppose

(i) $f(z)$ is analytic in a domain defined by $|z|<1$

(ii) $|f(z)| < 1$

(iii) $f(0)=0$

Then $|f(z)| \le |z|$ & $|f'(z)| < 1$

Equality holds only if $f(z)$ is linear transformation

$$w = f(z) = z e^{i\alpha} \quad ; \quad \alpha \in \mathbb{R}$$

Hence $|f'(0)| < 1$.

Problem:- $\int_C z^2 \cot \pi z \, dz = ?$ $\quad C: |z| = 2.5$

Solution:- $\int_C z^2 \cot \pi z \, dz = \frac{1}{\pi} \int_C z^2 \frac{\pi \cos \pi z}{\sin \pi z} dz$

$= \frac{1}{\pi} \int_C h(z) \cdot \frac{f'(z)}{f(z)} dz \qquad$ Here $h(z) = z^2$

$f(z) = \sin \pi z$

$= \frac{1}{\cancel{\pi}} \times 2\cancel{\pi} i \left[\sum n_i h(\alpha_i) \right]$

$= 2i \left[\sum n_i h(\alpha_i) \right]$

$f(z) = \sin \pi z = \sin 0$

$\cancel{\pi} z = n \cancel{\pi}$

$\Rightarrow z = n$

$\Rightarrow z = 0, 1, 2, -1, -2 \qquad C: |z| = 2.5$

$\int_C z^2 \cot \pi z \, dz = 2i \left[h(0) + h(1) + h(-1) + h(2) + h(-2) \right]$

$= 2i \left[0 + 1^2 + (-1)^2 + (2)^2 + (-2)^2 \right]$

$= 2i \times 10$

$= 20i$

i.e. $\boxed{\int_C z^2 \cot \pi z \, dz = 20i \quad ; C: |z| = 2.5}$ Ans

S.K.RATHORE

V.Imp. → $u(x,y) = \log(\sqrt{x^2+y^2})$ in $D = \{z : R_1 < |z| < R_2\}$

where $R_2 > R_1 \geq 0$ $\nexists$ a $v(x,y)$ s.t. $f = u + iv$

is analytic.

✓ $u(x,y) = \log(\sqrt{x^2+y^2})$ ही एक ऐसा Function है जिसके लिए कोई भी Harmonic conjugate exist नहीं करता है।

V.Imp → $f(z) = e^{2\pi i z} - 1$ this example shows that Mean Value Theorem is not true in Complex Analysis.

→ If $R = 0$ then $\sum_{n=0}^{\infty} a_n z^n$ converges for $z = 0$

V.Imp. → If $R = \infty$ then $\sum_{n=0}^{\infty} a_n z^n$ converge for all $z \in \mathbb{C}$.

→ If $0 < R < \infty$ then $\sum_{n=0}^{\infty} a_n z^n$ converge for $|z| < R$ & diverge for $|z| > R$.

$R.O.C = 1/R \leftarrow \sum_{0}^{\infty} a_n z^n$

→ $f(z) = \begin{cases} \dfrac{1}{\sin 1/z} & ; \quad 0 \leq z < 1 \\ 1 & ; \quad 1 \leq z < 2 \\ 3 & ; \quad 2 \leq z < 3 \end{cases}$

Infinite no. of non-isolated E.S & Infinite No. of Isolated singul.

Very Important Formulae

S.K. RATHORE

(i) $\displaystyle\int_0^{2\pi} \frac{d\theta}{a+b\sin\theta} = \frac{2\pi}{\sqrt{a^2-b^2}}$ if $a>0,\ b<0$ $\boxed{OR}$ $a<0,\ b>0$ $\boxed{OR}$ $a>0,\ b>0$

(ii) $\displaystyle\int_0^{2\pi} \frac{d\theta}{a+b\sin\theta} = -\frac{2\pi}{\sqrt{a^2-b^2}}$ if $a<0,\ b<0$.

(iii) $\displaystyle\int_0^{2\pi} \frac{d\theta}{(a+b\sin\theta)^2} = \frac{2\pi a}{(a^2-b^2)^{3/2}}$ if $a>0,\ b>0$ $\boxed{OR}$ $a<0,\ b<0$ $\boxed{OR}$ $a>0,\ b<0$

(iv) $\displaystyle\int_0^{2\pi} \frac{d\theta}{(a+b\sin\theta)^2} = \frac{-2\pi a}{(a^2-b^2)^{3/2}}$ if $a<0,\ b>0$

(v) $\displaystyle\int_0^{2\pi} \frac{d\theta}{(a+b\cos\theta)^2} = \frac{2\pi a}{(a^2-b^2)^{3/2}}$ $\forall$ $a>0,\ b>0$ $\boxed{OR}$ $a<0,\ b<0$ $\boxed{OR}$ $a>0,\ b<0$.

(vi) $\displaystyle\int_0^{2\pi} \frac{d\theta}{(a+b\cos\theta)^2} = -\frac{2\pi a}{(a^2-b^2)^{3/2}}$ $\forall$ $a<0,\ b>0$

(vii) $\displaystyle\int_0^{2\pi} \frac{d\theta}{a^2+\sin^2\theta} = \frac{\pi}{2a\sqrt{1+a^2}}$

(viii) $\displaystyle\int_0^{\pi} \frac{d\theta}{a^2+\sin^2\theta} = \frac{\pi}{a\sqrt{1+a^2}}$

(ix) $\displaystyle\int_0^{2\pi} \cos^{2n}\theta\, d\theta = \int_0^{2\pi} \sin^{2n}\theta\, d\theta = \frac{2\pi\, \lfloor 2n}{2^{2n}\,(\lfloor n)^2}$

(x) $\displaystyle\int_0^{2\pi} \frac{\sin^2\theta\, d\theta}{a+b\cos\theta} = \frac{2\pi}{b^2}\left\{a-\sqrt{a^2-b^2}\right\}$

(xi) $\displaystyle\int_0^{2\pi} (a\cos\theta+b\sin\theta)^{2n}\, d\theta = \frac{2\pi\, \lfloor 2n\, (a^2+b^2)^n}{2^{2n}\,(\lfloor n)^2}$

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(xii) $\int_0^{2\pi} \frac{\cos 2\theta \, d\theta}{1-2a\cos\theta+a^2} = \frac{\pi a^2}{2(1-a^2)}$

(xiii) $\int_0^{2\pi} e^{\cos\theta} \cos(n\theta - \sin\theta) \, d\theta = \frac{2\pi}{\lfloor n}$

(xiv) $\int_0^{\infty} \frac{x^{m-1}}{1+x^n} dx = \frac{\pi}{n\sin\left(\frac{m\pi}{n}\right)}$

(xv) $\int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2+1} dx = \frac{\pi}{2}(1-e^{-2})$

(xvi) $\int_{-\infty}^{\infty} \frac{\cos^2 x}{x^2+1} dx = \frac{\pi}{2}(1+3e^{-2})$

(xvii) $\int_0^{\infty} \frac{x\sin mx}{x^2+a^2} dx = \frac{\pi}{2a} e^{-ma}$; $a>0$

$= \frac{\pi}{2} e^{-ma}$; $a>0, m>0$.

(xviii) $\int_0^{\infty} \frac{\sin mx}{x(x^2+a^2)} = \frac{\pi}{2a^2}(1-e^{-ma})$

(xix) $\int_0^{\infty} \frac{\cos mx}{x^2+a^2} dx = \frac{\pi}{2a} e^{-ma}$.

(xx) $\int_0^{\infty} \frac{\cos mx}{(x^2+a^2)^2} dx = \frac{\pi}{4a^3} (1+ma) e^{-ma}$.

(xxi) $\int_{-\infty}^{\infty} \frac{\cos x \, dx}{(x^2+a^2)(x^2+b^2)} = \frac{\pi}{a^2-b^2}\left(\frac{e^{-b}}{b} - \frac{e^{-a}}{a}\right)$

(xxii) $\int_{-\infty}^{\infty} \frac{x\sin mx}{x^2-1} dx = \pi\cos m$.

(14a)

S.K.RATHORE

(xxiii) $\int_0^\infty e^{-x^2}\,dx = \sqrt{\pi}$

(xxiv) $\int_{-\infty}^{\infty} \frac{e^{ax}}{\cosh x}\,dx = \frac{\pi}{\cos\left(\frac{a\pi}{2}\right)}$